Welfare Assessments with Heterogeneous Individuals*

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Abstract

This paper introduces a decomposition of welfare assessments for general dynamic stochastic economies with heterogeneous individuals. The decomposition is based on constructing individual, dynamic, and stochastic weights that characterize how welfarist planners make tradeoffs across individuals, dates, and histories. Guided by the compensation principle, it initially decomposes a welfare assessment into an efficiency and a redistribution component, while the efficiency component is further decomposed into i) aggregate efficiency, ii) risk-sharing, and iii) intertemporal-sharing components. Five minimal examples and three applications illustrate the properties of the decomposition and how it can be used to draw normative conclusions in specific scenarios.

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Keywords: welfare decomposition, welfare assessments, heterogeneous agents, incomplete markets, interpersonal welfare comparisons, social welfare functions

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1 Introduction

Assessing the aggregate welfare impact of policies or shocks in dynamic stochastic economies with heterogeneous individuals and imperfect financial markets is far from trivial. In particular, it is challenging to identify the specific normative considerations that underlie a particular welfare assessment. This paper tackles this challenge by developing a decomposition of welfare assessments based on individual, dynamic, and stochastic weights that satisfies desirable properties.

We introduce our results in a canonical dynamic stochastic environment in which heterogeneous individuals consume a single good and supply a single factor (labor) at each history. We consider welfare assessments for welfarist planners — those who use a social welfare function. Since comparisons in utils are meaningless — due to the ordinal nature of individual utilities — we first express welfare assessments in terms of normalized individual, dynamic, and stochastic weights, which allow us to interpret how welfarist planners make tradeoffs across individuals, dates, and histories in common units. To define such units, we select lifetime, date, and history welfare numeraires.

After expressing welfare assessments in comparable units, we show how to decompose a welfare assessment into i) an efficiency component and ii) a redistribution component, as illustrated by Figure 1. We treat as an *axiom* that the efficiency component of our decomposition must satisfy the *compensation principle* (Boadway and Bruce, 1984; Feldman, 1998). That is, we want the efficiency component to represent the net gain of the perturbation once the winners have hypothetically compensated the losers if transfers were feasible and costless. Therefore, the efficiency component of our decomposition necessarily corresponds to Kaldor-Hicks efficiency: it is the sum of individual willingness-to-pay for the perturbation in units of the lifetime welfare numeraire. We show that there is a unique decomposition in which a normalized welfare assessment can be expressed as the sum of a component that satisfies the compensation principle (efficiency) and its complement (redistribution). The redistribution component, which captures the equity concerns embedded in a particular social welfare function, is positive when those individuals relatively favored in a perturbation are those relatively preferred by the planner, i.e., have higher normalized individual weights.

We establish three properties of the efficiency/redistribution decomposition. First, the efficiency component is identical for all welfarist planners, which implies that all differences in the assessments of welfarist planners are due to redistribution considerations. Second, the efficiency component is invariant to preference-preserving utility transformations, which implies that the impact of preference-preserving utility transformations on welfare assessments is exclusively confined to the redistribution component. Third, anything goes for the redistribution component. That is, there are social welfare functions and preference-preserving utility transformations such that redistribution can be positive or negative for the same perturbation.

By appealing once again to the compensation principle — now at each date and history —

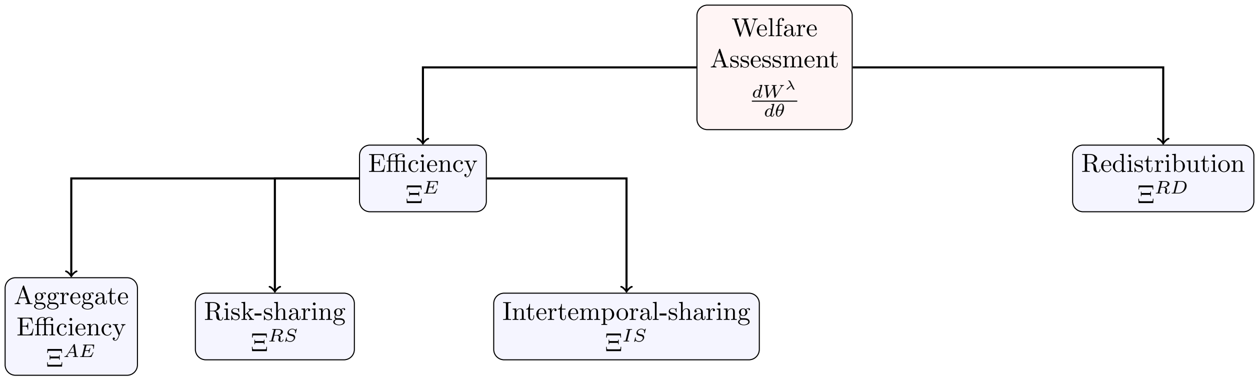

Figure 1: Welfare Decomposition

Note: This figure illustrates the decomposition of welfare assessments introduced in this paper. Proposition 1 introduces the efficiency/redistribution decomposition and Proposition 3 introduces the aggregate efficiency/risk-sharing/intertemporal-sharing decomposition. See Propositions 2, 4, 5, 6, and 7 for properties.

we decompose efficiency gains into a component that captures changes in aggregate history welfare gains (aggregate efficiency) and two components that capture the differential impact of a perturbation towards individuals with different valuations across dates (intertemporal-sharing) and histories (risk-sharing). For given welfare numeraires, this is the unique decomposition in which the efficiency component can be expressed as the discounted sum — using aggregate time and stochastic discount factors — of aggregate history welfare gains and its complement.

The differences in individual valuations across dates and histories that govern the risk-sharing and intertemporal-sharing components depend on the extent to which individuals can freely smooth consumption (in general, the history welfare numeraire) intertemporally and across histories. We hence show that i) the risk-sharing and intertemporal-sharing components are zero when marginal rates of substitution across all dates and histories are equalized across individuals — a condition that complete markets economies satisfy — and ii) the intertemporal-sharing component is zero when marginal rates of substitution across dates are equalized across individuals — a condition satisfied when all individuals frictionlessly borrow and save.

More generally, we identify conditions on i) normalized weights and ii) welfare gains that guarantee that the risk-sharing, intertemporal-sharing, or redistribution components are zero. Intuitively, normalized weights and welfare gains must vary cross-sectionally along the relevant dimensions for these three components to be non-zero. We also identify particular economies of practical relevance in which specific components of the welfare decomposition are zero. We show that i) single individual economies exclusively feature aggregate efficiency, ii) risk-sharing is zero in perfect foresight economies, iii) intertemporal-sharing and redistribution are zero in economies with ex-ante (but not necessarily ex-post) identical individuals, iv) risk- and intertemporal-sharing are zero in static economies, v) aggregate efficiency is zero in single good endowment economies in which the aggregate endowment is fixed. We also characterize which particular components of the welfare decomposition are zero when planners can costlessly and optimally transfer resources among individuals along particular dimensions.

Five minimal examples and three applications of practical relevance illustrate the properties of the decomposition and how it can be used to draw normative conclusions. Our first application analyzes the welfare effects of a transfer policy that smooths consumption across individuals who face idiosyncratic consumption risk. The central takeaway is that the persistence of the endowment process determines whether welfare gains are attributed to risk-sharing, intertemporal-sharing, or redistribution. Our results also highlight that a flat term structure of welfare assessments may mask substantial time variation on each of component of the welfare decomposition, with backloaded risk-sharing gains and frontloaded intertemporal-sharing and redistribution gains that turn into losses in the long run.

Our second application contrasts the welfare effects of (linear) labor income taxes in two settings: i) a deterministic environment in which individuals differ in their productivity at the time of the welfare assessment, and ii) a stochastic environment in which individuals are identical at the time of the welfare assessment, but experience different shocks. In both environments, increasing tax rates causes aggregate efficiency losses by distorting labor supply. While both environments can be parameterized to yield a quantitatively identical optimal tax, a utilitarian planner attributes the welfare gains from the tax to redistribution in the deterministic environment and to risk-sharing in the stochastic environment. Moreover, in the stochastic environment *all* welfarist planners agree on the magnitude of the optimal tax, which is Pareto-improving in that case, while in the deterministic environment the optimal tax is sensitive to the choice of social welfare function. This application also illustrates that perturbations may yield efficiency gains even though aggregate consumption falls at all times.

Our third application studies the welfare implications of a change in credit conditions in an economy in which borrowing-constrained individuals make an investment decision. Considering changes in the borrowing limit in this economy is a tractable perturbation that parameterizes changes in the degree of market completeness. This application illustrates how relaxing a borrowing constraint can feature at the same time i) positive aggregate efficiency and intertemporal-sharing components, by allowing investors to invest more and by reallocating resources towards borrowing-constrained individuals, and ii) a negative risk-sharing component, since investors end up bearing higher risk by virtue of their increased investment. This application also illustrates how the redistributive implications of a change in credit conditions can i) be traced back to pecuniary effects in competitive economies and ii) vary depending on the level of the borrowing limit.

Finally, Section 5 briefly summarizes extensions and additional results covered in the Online Appendix. There, we describe how to leverage the welfare decomposition to systematically construct non-welfarist welfare criteria based on individual, dynamic, and stochastic weights (DS-planners), extend our approach to more general environments, and show how to further decompose the components of the welfare decomposition, among other results.

Related Literature. This paper contributes to several literatures, including those on i) welfare decompositions, ii) welfare evaluation in dynamic stochastic environments, iii) interpersonal welfare comparisons, and iv) institutional mandates.

This paper is by no means the first to think of identifying or decomposing the contributions of policy or parameter changes to social welfare. The seminal paper in this literature is Benabou (2002), who proposes i) a measure of aggregate economic efficiency based on computing and aggregating certainty-equivalent consumption levels, and ii) a new index of social welfare, which also accounts for redistribution.[1] Floden (2001) further advances this certainty-equivalent approach by decomposing aggregate economic efficiency into welfare gains of increased levels and welfare gains of reduced uncertainty. Variations and enhancements of this approach have been used in a variety of contexts, for instance, by Seshadri and Yuki (2004), Conesa, Kitao and Krueger (2009), Koehne and Kuhn (2015), Dyrda and Pedroni (2023), and Bhandari et al. (2021), among others.

While our decomposition shares the motivation for decomposing welfare gains with existing decompositions — understanding the role of missing markets or equity concerns — there is no formal relation between existing decompositions and ours, as implied by the properties of our decomposition established in Section 3. We highlight such differences by contrasting our results mainly to those of Bhandari et al. (2021), whose decomposition refines that of Benabou (2002) and Floden (2001), but applies to environments with general preferences and social welfare functions. Bhandari et al. (2021) decompose welfare changes into three components: a first component captures changes in aggregate consumption, a second component (redistribution) captures changes in expected consumption shares, and a third component (insurance) captures changes in the innovation to consumption shares. Their first component may seem heuristically related to our aggregate efficiency component, the second component to our redistribution component, and their third component to our paper's risk-sharing and intertemporal-sharing components. However, besides the resemblance in labels, there are no formal similarities between the decompositions.

We highlight three significant differences. First and most importantly, the counterparts in existing decompositions of our efficiency component are not invariant to preference-preserving transformations. In an attempt to sidestep this issue, earlier work such as Benabou (2002) and Floden (2001) restricts the scope of their decompositions to models in which all individuals have identical utility functions. In later work, such as Bhandari et al. (2021), all three elements of

[1] As the measures of efficiency in Benabou (2002) and subsequent work differ from Kaldor-Hicks efficiency, our efficiency and redistribution components necessarily differ from those in existing decompositions.

their decomposition are sensitive to preference-preserving utility transformations or the choice of social welfare function. This contrasts with the invariance result for the efficiency component (and consequently the aggregate efficiency, risk-sharing, and intertemporal-sharing components) of our decomposition. Second, the risk-sharing and intertemporal-sharing components in our decomposition can only be non-zero when markets are incomplete, while existing decompositions would attribute non-zero welfare gains to their insurance component in complete market or representative agent economies. Finally, the properties for perturbations with specific consumption changes established by Bhandari et al. (2021) neither imply nor are implied by the properties in Proposition 5a). This partly occurs because Bhandari et al. (2021) consider proportional changes while Proposition 5a) considers changes in levels. Section J of the Online Appendix further explains how our results relate to existing work.

Our results most directly build on Alvarez and Jermann (2004)'s marginal reformulation of the consumption-equivalent approach introduced by Lucas (1987). Their paper develops the use of marginal methods to think about welfare assessments in the representative agent case, and our results can be interpreted as generalizing their approach to the case with heterogeneous agents. We show that our results nest those of Alvarez and Jermann (2004) in Section J.2 of the Online Appendix.

The question of how to make interpersonal welfare comparisons to form aggregate welfare assessments has a long history in economics — see, among many others, Kaldor (1939), Hicks (1939), Bergson (1938), Samuelson (1947), Harsanyi (1955), Sen (1970) or, more recently, Kaplow and Shavell (2001), Saez and Stantcheva (2016), Hendren (2020), Hendren and Sprung-Keyser (2020), and Schulz, Tsyvinski and Werquin (2023). However, perhaps surprisingly, dynamic stochastic considerations have not been central to this literature. By introducing normalized weights, our results provide a new characterization of how welfarist planners make tradeoff across individuals, dates, and histories.

At last, by allowing for welfare criteria based not only on individual generalized weights, but also dynamic and stochastic generalized weights, we generalize the work of Saez and Stantcheva (2016) in Section J.1 of the Online Appendix. Those results open the door to future disciplined discussions on policy-making mandates, for instance, by justifying and defining institutional mandates that incorporate or disregard specific cross-sectional considerations, such as risk-sharing, intertemporal-sharing, or redistribution.

2 Environment

Our notation closely follows that of Chapter 8 of Ljungqvist and Sargent (2018). We consider an economy populated by a finite number $I \geq 1$ of individuals, indexed by $i \in \mathcal{I} = \{1, \ldots, I\}$. At each date $t \in \{0, \ldots, T\}$, where $0 \leq T \leq \infty$, there is a realization of a stochastic event $s_t \in S$. We denote the history of events up to date t by $s^t = (s_0, s_1, \ldots, s_t)$, and the probability of observing a particular sequence of events s^t by $\pi_t(s^t)$. The initial value of s_0 is predetermined, so $\pi_0(s_0) = 1$. At all dates and histories, individuals consume a single good and supply a single factor, e.g. labor.

Preferences. An individual i derives utility from consumption and (dis)utility from factor supply, with a lifetime utility representation given by

$$V^i = \sum_t \left(\beta^i\right)^t \sum_{s^t} \pi_t\left(s^t\right) u_t^i\left(c_t^i\left(s^t\right), n_t^i\left(s^t\right); s^t\right), \qquad \text{(Preferences)} \tag{1}$$

where $c_t^i(s^t)$ and $n_t^i(s^t)$ respectively denote the consumption and factor supply of individual i at history s^t. We denote individual i's instantaneous utility at history s^t by $u_t^i(\cdot; s^t)$, where $\frac{\partial u_t^i(s^t)}{\partial c_t^i} := \frac{\partial u_t^i(c_t^i(s^t), n_t^i(s^t); s^t)}{\partial c_t^i(s^t)} > 0$ and $\frac{\partial u_t^i(s^t)}{\partial n_t^i} := \frac{\partial u_t^i(c_t^i(s^t), n_t^i(s^t); s^t)}{\partial n_t^i(s^t)} < 0$, and where Inada conditions apply to both consumption and factor supply. We denote individual i's discount factor by $\beta^i \in [0, 1)$. We refer to the unit of V^i as individual i utils.

Equation (1) corresponds to the time-separable expected utility preferences with exponential discounting and homogeneous beliefs widely used in macroeconomics and finance, augmented to allow for time- and history-dependent individual-specific preferences. Section G of the Online Appendix considers more general environments.

Perturbation. We assume that $c_t^i(s^t)$ and $n_t^i(s^t)$ are smooth functions of a perturbation parameter $\theta \in [0, 1]$, so derivatives such as $\frac{dc_t^i(s^t)}{d\theta}$ and $\frac{dn_t^i(s^t)}{d\theta}$ are well-defined. A perturbation $d\theta$ may capture changes in policies or any other primitive in a fully specified model. Typically, the mapping between consumption and factor supply, $c_t^i(s^t)$ and $n_t^i(s^t)$, and θ — which we take as given — emerges endogenously and accounts for general equilibrium effects, as we illustrate in our applications. However, our results do not require to further specify technologies, resource or budget constraints, equilibrium notions, etc. Alternatively, a perturbation may capture changes in allocations directly chosen by a planner.

Social Welfare Function. We study welfare assessments for welfarist planners, that is, planners with a social welfare function given by

$$W = \mathcal{W}\left(V^1, \ldots, V^i, \ldots, V^I\right), \qquad \text{(Social Welfare Function)} \tag{2}$$

where individual lifetime utilities V^i are defined in (1).[2] We refer to the units of W as social utils. In the body of the paper, we assume that $\frac{\partial \mathcal{W}}{\partial V^i} > 0$, $\forall i$. Section G.5 of the Online Appendix allows for $\frac{\partial \mathcal{W}}{\partial V^i} = 0$ for some individuals and Section E considers non-welfarist welfare criteria.

A welfarist planner finds a perturbation $d\theta$ desirable (undesirable) if

$$\frac{dW}{d\theta} = \sum_i \frac{\partial \mathcal{W}}{\partial V^i} \frac{dV^i}{d\theta} > (<) \, 0.$$

The welfarist approach is widely used because it is Paretian, that is, it concludes that every Pareto-improving perturbation is desirable.[3] However, because individual utilities are ordinal, understanding how a welfarist planner makes tradeoffs in comparable units is not straightforward, as we show next.

3 Welfare Decomposition

In this section, we present the paper's central result: a decomposition of welfare assessments for welfarist planners that satisfies desirable properties.

3.1 Normalized Welfare Assessment

In order to introduce the decomposition, it is first necessary to understand how a welfarist planner values welfare gains across individuals, dates, and histories. Since comparisons in utils are meaningless — due to the ordinal nature of individual utilities — we choose comparable units to express welfare gains. We refer to these units as lifetime, date, and history welfare numeraires. Lemma 1 thus represents welfare assessments in terms of normalized welfare gains and normalized weights. Normalized welfare gains represent lifetime, date, and history welfare gains for different individuals in welfare numeraire units. Normalized weights capture how welfarist planners make tradeoffs in such common units.

Goods or factors (or bundles of goods or factors) that can be easily transferred — at least hypothetically — across individuals either privately or by a planner are natural welfare numeraires since they justify the use of the compensation principle. See Propositions 1 and 3 for applications

[2] As in Boadway and Bruce (1984) or Kaplow (2011), we refer to the use of social welfare functions — typically traced back to Bergson (1938) and Samuelson (1947) — as the *welfarist* approach. As explained in Kaplow (2011), the critical restriction implied by the welfarist approach is that the social welfare function $\mathcal{W}(\cdot)$ cannot depend on any model outcomes besides individual utility levels. The utilitarian social welfare function, which adds up a (weighted) sum of individual utilities, is the most used in practice. See Mas-Colell, Whinston and Green (1995), Kaplow (2011), or Adler and Fleurbaey (2016) for descriptions of alternative social welfare functions.

[3] Kaplow and Shavell (2001) show that the converse statement — welfarism is the only approach that respects the Pareto criterion — is true under minimal assumptions. A perturbation is strictly (weakly) Pareto-improving if every individual i is strictly (weakly) better off after the change. Formally, a perturbation is strictly (weakly) Pareto-improving when $\frac{dV^i}{d\theta} > (\geq) \, 0$, $\forall i$. Even though Pareto improvements are unambiguously desirable, they are rare in economies with many individuals.

of this principle. Therefore, in economies with a single consumption good like the one considered here, it is natural to aggregate and compare welfare gains in consumption-equivalents. That is, it is natural to choose a unit of the consumption good as the history welfare numeraire at each history (history-s^t consumption), a unit of the consumption good at all histories at a given date as the date welfare numeraire at each date (date-t consumption), and a unit of the consumption good at all dates and histories as the lifetime welfare numeraire (perpetual consumption). Hence, to simplify the exposition, we adopt such unit-consumption-based welfare numeraires in the body of the paper. Section F of the Online Appendix considers general welfare numeraires.

Given the choice of unit-consumption-based welfare numeraires, Lemma 1 expresses welfare assessments in terms of the inputs of the components of the welfare decomposition: normalized lifetime, date, and history welfare gains, and normalized individual, dynamic, and stochastic weights. In terms of notation, variables with a λ superindex are expressed in the appropriate numeraire.[4]

Lemma 1. *(Normalized Welfare Gains and Normalized Weights) A normalized welfare assessment for a welfarist planner can be represented as*

$$\frac{dW^\lambda}{d\theta} = \frac{\frac{dW}{d\theta}}{\frac{1}{I}\sum_i \frac{\partial \mathcal{W}}{\partial V^i} \sum_t \left(\beta^i\right)^t \sum_{s^t} \pi_t\left(s^t\right) \frac{\partial u_t^i(s^t)}{\partial c_t^i}} = \sum_i \omega^i \frac{dV^{i|\lambda}}{d\theta}, \tag{3}$$

where $\frac{dV^{i|\lambda}}{d\theta}$, $\frac{dV_t^{i|\lambda}}{d\theta}$, *and* $\frac{dV_t^{i|\lambda}(s^t)}{d\theta}$ *respectively denote lifetime, date, and history welfare gains, given by*

$$\frac{dV^{i|\lambda}}{d\theta} = \sum_t \omega_t^i \frac{dV_t^{i|\lambda}}{d\theta} \qquad \text{(Normalized Lifetime Welfare Gains)} \tag{4}$$

$$\frac{dV_t^{i|\lambda}}{d\theta} = \sum_{s^t} \omega_t^i \left(s^t\right) \frac{dV_t^{i|\lambda}\left(s^t\right)}{d\theta} \qquad \text{(Normalized Date Welfare Gains)} \tag{5}$$

$$\frac{dV_t^{i|\lambda}\left(s^t\right)}{d\theta} = \frac{dc_t^i\left(s^t\right)}{d\theta} + \frac{\frac{\partial u_t^i(s^t)}{\partial n_t^i}}{\frac{\partial u_t^i(s^t)}{\partial c_t^i}} \frac{dn_t^i\left(s^t\right)}{d\theta}. \qquad \text{(Normalized History Welfare Gains)} \tag{6}$$

And ω^i, ω_t^i, *and* $\omega_t^i\left(s^t\right)$ *respectively denote normalized individual, dynamic, and stochastic weights,*

[4] We use the superindex λ to denote normalized gains because in Section F of the Online Appendix we denote the triple of normalizing factors (lifetime, date, and history) that allow for general welfare numeraires by λ^i, λ_t^i, and $\lambda_t^i\left(s^t\right)$.

given by

$$\omega^{i}=\frac{\frac{\partial\mathcal{W}}{\partial V^{i}}\sum_{t}\left(\beta^{i}\right)^{t}\sum_{s^{t}}\pi_{t}\left(s^{t}\right)\frac{\partial u_{t}^{i}\left(s^{t}\right)}{\partial c_{t}^{i}}}{\frac{1}{I}\sum_{i}\frac{\partial\mathcal{W}}{\partial V^{i}}\sum_{t}\left(\beta^{i}\right)^{t}\sum_{s^{t}}\pi_{t}\left(s^{t}\right)\frac{\partial u_{t}^{i}\left(s^{t}\right)}{\partial c_{t}^{i}}}\qquad\text{(Normalized Individual Weight)}\qquad(7)$$

$$\omega_{t}^{i}=\frac{\left(\beta^{i}\right)^{t}\sum_{s^{t}}\pi_{t}\left(s^{t}\right)\frac{\partial u_{t}^{i}\left(s^{t}\right)}{\partial c_{t}^{i}}}{\sum_{t}\left(\beta^{i}\right)^{t}\sum_{s^{t}}\pi_{t}\left(s^{t}\right)\frac{\partial u_{t}^{i}\left(s^{t}\right)}{\partial c_{t}^{i}}}\qquad\text{(Normalized Dynamic Weight)}\qquad(8)$$

$$\omega_{t}^{i}\left(s^{t}\right)=\frac{\pi_{t}\left(s^{t}\right)\frac{\partial u_{t}^{i}\left(s^{t}\right)}{\partial c_{t}^{i}}}{\sum_{s^{t}}\pi_{t}\left(s^{t}\right)\frac{\partial u_{t}^{i}\left(s^{t}\right)}{\partial c_{t}^{i}}}.\qquad\text{(Normalized Stochastic Weight)}\qquad(9)$$

Normalized lifetime welfare gains for individual i, $\frac{dV^{i|\lambda}}{d\theta}$, have the interpretation of i's willingness-to-pay for the perturbation in units of the lifetime welfare numeraire (a unit of perpetual consumption). Normalized date welfare gains for individual i at date t, $\frac{dV_{t}^{i|\lambda}}{d\theta}$, correspond to i's willingness-to-pay for the perturbation at date t in units of the date welfare numeraire (a unit of date-t consumption). Normalized history welfare gains for individual i at history s^t, $\frac{dV_{t}^{i|\lambda}\left(s^{t}\right)}{d\theta}$, correspond to i's willingness-to-pay for the perturbation at history s^t in units of the history welfare numeraire (a unit of history-s^t consumption).

Normalized history welfare gains, $\frac{dV_{t}^{i|\lambda}\left(s^{t}\right)}{d\theta}$, define a consumption-equivalent at a particular history, while date and lifetime gains can be interpreted as time- and risk-discounted sums of normalized history welfare gains. In fact, Lemma 1 shows that every welfare assessment can be expressed as a triple weighted sum of normalized history welfare gains, since

$$\frac{dW^{\lambda}}{d\theta}=\sum_{i}\omega^{i}\sum_{t}\omega_{t}^{i}\sum_{s^{t}}\omega_{t}^{i}\left(s^{t}\right)\frac{dV_{t}^{i|\lambda}\left(s^{t}\right)}{d\theta}.$$

Dividing $\frac{dW}{d\theta}$ by $\frac{1}{I}\sum_{i}\frac{\partial\mathcal{W}}{\partial V^{i}}\sum_{t}\left(\beta^{i}\right)^{t}\sum_{s^{t}}\pi_{t}\left(s^{t}\right)\frac{\partial u_{t}^{i}\left(s^{t}\right)}{\partial c_{t}^{i}}$ ensures that the normalized welfare assessment $\frac{dW^{\lambda}}{d\theta}$ is expressed in units of the lifetime welfare numeraire, and that it can be interpreted in terms of a perturbation that distributes perpetual consumption equally. That is, a normalized welfare assessment of, for instance, $\frac{dW^{\lambda}}{d\theta}=3$ is equivalent to a perturbation in which 3 units of perpetual consumption are equally distributed across all individuals. Unnormalized and normalized assessments, $\frac{dW}{d\theta}$ and $\frac{dW^{\lambda}}{d\theta}$, agree on whether a perturbation is desirable or not.

The normalized individual weight ω^i defines how a welfarist planner trades off lifetime welfare gains across individuals. For instance, if $\omega^i = 1.3$, a welfarist planner finds the welfare gain from giving 1 unit of perpetual consumption to individual i equivalent to giving 1.3 units equally across all individuals. Note that normalized individual weights average to one, so $\frac{1}{I}\sum_{i}\omega^{i}=1$.

The normalized dynamic weight ω_t^i defines a marginal rate of substitution between a unit of date-t consumption and a unit of perpetual consumption for individual i. For instance, if $\omega_t^i = 0.1$, a welfarist planner finds the welfare gain from giving 1 unit of date-t consumption-equivalent to individual i equivalent to giving 0.1 units of perpetual consumption to that individual. Since perpetual consumption is a bundle of consumption at all dates, normalized dynamic weights add up to one, defining a *normalized discount factor*, so $\sum_t \omega_t^i = 1$, $\forall i$.

The normalized stochastic weight $\omega_t^i(s^t)$ defines a marginal rate of substitution between a unit of history-s^t consumption and a unit of date-t consumption for individual i. For instance, if $\omega_t^i(s^t) = 0.4$, a welfarist planner finds the welfare gain from giving 1 unit of history-s^t consumption-equivalent to individual i equivalent to giving 0.4 units of date-t consumption to that individual. Normalized stochastic weights add up to one, so $\sum_{s^t} \omega_t^i(s^t) = 1$, $\forall t$, $\forall i$, defining *risk-neutral probabilities*.[5]

Through the lens of asset pricing, Lemma 1 implies that every welfare assessment corresponds to a weighted sum of the values given by I individuals to claims to the normalized history welfare gains, $\frac{dV_t^{i|\lambda}(s^t)}{d\theta}$, which play the role of individual-specific payoffs. This is related to but different from the literature on asset pricing with incomplete markets (Constantinides and Duffie, 1996; Krueger and Lustig, 2010), which focuses on pricing claims in which payoffs are not individual-specific.

3.2 Efficiency vs. Redistribution

After expressing welfare assessments in comparable units, any decomposition of welfare assessments corresponds to a particular grouping of the terms that define $\frac{dW^\lambda}{d\theta}$ in Lemma 1. First, we seek to decompose a normalized welfare assessment $\frac{dW^\lambda}{d\theta}$ into an efficiency component Ξ^E and a redistribution component Ξ^{RD}.

We will treat as an *axiom* that the efficiency component of our decomposition must satisfy the *compensation principle* (Boadway and Bruce, 1984; Feldman, 1998). That is, we want Ξ^E to represent the net gain in terms of the lifetime welfare numeraire once the winners of a perturbation have hypothetically compensated the losers if transfers were feasible and costless. Therefore, the efficiency component of our decomposition necessarily corresponds to Kaldor-Hicks efficiency (Kaldor, 1939; Hicks, 1939) — defined as the sum of individual willingness-to-pay for the perturbation in units of the lifetime welfare numeraire. Consequently, perturbations in which $\Xi^E > 0$ could be turned into Pareto improvements if transfers were feasible and costless.

Proposition 1 shows that there is a *unique* way to decompose a normalized welfare assessment into i) an *efficiency* component that satisfies the compensation principle by adding up normalized lifetime welfare gains across individuals (Kaldor-Hicks efficiency), and ii) its complement, a *redistribution* component, which captures the differential impact of a perturbation towards those individuals

[5] Risk-neutral probabilities are widely used in finance (Duffie, 2001; Cochrane, 2005), while normalized discount factors are common in the study of repeated games (Fudenberg and Tirole, 1991; Mailath and Samuelson, 2006).

preferred by the welfarist planner.

Proposition 1. *(Efficiency/Redistribution Decomposition) A normalized welfare assessment for a welfarist planner can be decomposed into efficiency and redistribution components, Ξ^E and Ξ^{RD}, as follows:*

$$\underbrace{\frac{dW^\lambda}{d\theta}}_{\substack{\textit{Welfare}\\ \textit{Assessment}}} = \sum_i \omega^i \frac{dV^{i|\lambda}}{d\theta} = \underbrace{\sum_i \frac{dV^{i|\lambda}}{d\theta}}_{\Xi^E \ (\textit{Efficiency})} + \underbrace{\mathbb{C}ov_i^\Sigma\left[\omega^i, \frac{dV^{i|\lambda}}{d\theta}\right]}_{\Xi^{RD} \ (\textit{Redistribution})}, \tag{10}$$

where $\mathbb{C}ov_i^\Sigma[\cdot,\cdot] = I \cdot \mathbb{C}ov_i[\cdot,\cdot]$ denotes a cross-sectional covariance-sum among individuals. For a given lifetime welfare numeraire, this is the unique decomposition in which a normalized welfare assessment can be expressed as Kaldor-Hicks efficiency — that is, the unweighted sum of individual willingness-to-pay, which satisfies the compensation principle — and its complement.

Proposition 1 implies that welfare assessments based on a social welfare function are equivalent to relying on Kaldor-Hicks efficiency with a correction for redistribution. That is, this result shows that maximizing a social welfare function is identical to maximizing Kaldor-Hicks efficiency with a redistribution correction.

Two properties relate the efficiency component (equivalently, Kaldor-Hicks efficiency) and Pareto efficiency, justifying our notion of efficiency based on the compensation principle. First, the efficiency component is strictly positive ($\Xi^E > 0$) for (strict or weak) Pareto-improving perturbations. Intuitively, since Pareto-improving perturbations feature no losers, the sum of willingness-to-pay must be strictly positive. Second, Pareto optimal allocations — defined as those solving the Pareto problem (Ljungqvist and Sargent, 2018) — must feature a weakly negative efficiency component ($\Xi^E \leq 0$) for any feasible perturbation given endowments and technologies, as shown in Section I.1 of the Online Appendix.[6] This property ensures that the efficiency component of our decomposition cannot be positive by reallocating resources away from a Pareto optimal allocation.[7]

The redistribution component — which can equivalently be expressed as $\Xi^{RD} = \sum_i \left(\omega^i - 1\right) \frac{dV^{i|\lambda}}{d\theta}$ — captures the equity concerns embedded in a particular social welfare function. Ξ^{RD} is positive when the individuals relatively favored in a perturbation are those relatively preferred by the planner, i.e., have higher normalized individual weights ω^i, as we illustrate in Examples 1 and 2 below.

3.2.1 Properties of Efficiency/Redistribution Decomposition

In addition to the properties of Kaldor-Hicks efficiency already discussed, Proposition 2 presents three properties of the efficiency/redistribution decomposition just introduced that further justify

[6] Perturbations that increase good or factor endowments or improve technologies can be Pareto improvements over allocations that were originally Pareto efficient given the initial endowments and technologies.

[7] The critical feature of the Pareto problem is the presence of linear resource constraints, which allow for costless transfers. In general, it is possible to find perturbations of *constrained* Pareto efficient allocations such that $\Xi^E > 0$.

the choice of labels for each component.

Proposition 2. *(Properties of Efficiency/Redistribution Decomposition)*

a) (Invariance of efficiency component to social welfare function) The efficiency component is identical for all welfarist planners. Differences in welfare assessments among welfarist planners are exclusively due to the redistribution component.

b) (Invariance of efficiency component to preference-preserving utility transformations) The efficiency component is invariant to i) monotonically increasing transformations of individuals' lifetime utilities and ii) positive affine (increasing linear) transformations of individuals' instantaneous utilities.

c) (Anything goes for redistribution) Whenever $I > 1$, there exist social welfare functions and preference-preserving utility transformations such that i) $\Xi^{RD} > 0$ and ii) $\Xi^{RD} < 0$ for any perturbation in which $dV^{i|\lambda} \neq 0$ for at least one individual.

Proposition 2a) follows from the fact that normalized lifetime utilities $\frac{dV^{i|\lambda}}{d\theta}$ do not depend on the choice of SWF; only the normalized individual weights ω^i do. This property implies that welfarist planners cannot disagree about the efficiency consequences of a perturbation. The reason why different welfarist planners make different welfare assessments is simply because they use different normalized individual weights, implying different social preferences for redistribution.

Proposition 2b) follows from the fact that normalized lifetime utilities $\frac{dV^{i|\lambda}}{d\theta}$ — with units $\frac{\text{lifetime welfare numeraire}}{\text{units of } \theta}$ — do not depend on the choice of individual utility units. This property implies that even though welfarist planners mechanically overweight welfare gains by individuals whose lifetime (instantaneous) utility experiences a monotonically increasing (positive affine) transformation (Campbell, 2018) — even though this has no impact on allocations — this only impacts the redistribution component.[8] Hence, the impact of preference-preserving utility transformations on welfare assessments is exclusively confined to the redistribution component.

Proposition 2c) highlights that as long as one individual does relatively better than another, it is possible to select individual weights ω^i (by varying the social welfare function or the formulations of individual utilities) so that $\mathbb{C}ov_i^{\Sigma}\left[\omega^i, \frac{dV^{i|\lambda}}{d\theta}\right]$ is positive or negative for a given perturbation. For instance, Ξ^{RD} can be negative for Pareto-improving perturbations, even though $\Xi^E + \Xi^{RD} > 0$ in that case since welfarist planners are Paretian, as we illustrate next.

3.2.2 Illustration of Efficiency/Redistribution Decomposition

Here, we present two minimal examples that illustrate the Efficiency/Redistribution Decomposition. These examples show how the redistribution term can have different signs for the same perturbation

[8]The invariance to positive affine transformations is only meaningful with expected utility preferences.

depending on the choice of social welfare function and utility units for purely redistributive perturbations (Example 1) and for Pareto-improving perturbations (Example 2).

Example 1. **[$\Xi^E = 0$ but $\Xi^{RD} \gtreqless 0$].** Consider a static deterministic economy with two individuals ($I = 2$), each endowed with one unit of consumption, with preferences $u^i\left(c^i\right)$. Consider a perturbation that transfers θ units of consumption from A to B according to

$$c^A = 1 + \theta \qquad \text{and} \qquad c^B = 1 - \theta.$$

This perturbation can be interpreted as a redistributive policy. Using consumption as numeraire, it trivially follows that $\Xi^E = \sum_i \frac{dV^{i|\lambda}}{d\theta} = \sum_i \frac{dc^i}{d\theta} = 0$, so the welfare assessment of this perturbation is exclusively driven by Ξ^{RD}. In this perturbation, A gains and B loses since

$$\frac{dV^{A|\lambda}}{d\theta} > 0 > \frac{dV^{B|\lambda}}{d\theta}.$$

Assuming a utilitarian social welfare function with Pareto weights given by α^i, the value of the redistribution component can be positive, negative, or zero, depending on $\omega^i = \alpha^i \frac{\partial u^i\left(c^i\right)}{\partial c^i} / \frac{1}{I} \sum_i \alpha^i \frac{\partial u^i\left(c^i\right)}{\partial c^i}$. If the individual weight for individual A is higher than for B, $\omega^A > \omega^B$, and $\Xi^{RD} > 0$. If instead $\omega^A < \omega^B$, then $\Xi^{RD} < 0$.

Example 2. **[Pareto Improvement with $\Xi^{RD} \gtreqless 0$].** Consider the same static deterministic economy as in Example 1. Consider now a perturbation that increases the consumption of both A and B according to

$$c^A = 1 + 2\theta \quad \text{and} \quad c^B = 1 + \theta.$$

This perturbation can be interpreted as a positive technology shock that favors some individuals more than others. It trivially follows that $\Xi^E = \sum_i \frac{dV^{i|\lambda}}{d\theta} = 3 > 0$. In this case, all individuals gain from the perturbation since

$$\frac{dV^{A|\lambda}}{d\theta} > \frac{dV^{B|\lambda}}{d\theta} > 0,$$

so this perturbation is a Pareto improvement. Assuming a utilitarian social welfare function with Pareto weights given by α^i, the value of the redistribution component can be positive, negative, or zero, depending once again on ω^i. If the individual weight for individual A is higher than for B, $\omega^A > \omega^B$, and $\Xi^{RD} > 0$. If instead $\omega^A < \omega^B$, then $\Xi^{RD} < 0$.

3.3 Aggregate Efficiency vs. Risk-Sharing vs. Intertemporal-Sharing

Proposition 1 implies that the efficiency component of a normalized welfare assessment is simply the sum of discounted individual welfare gains using individual discount factors — captured by normalized dynamic and stochastic weights. Hence, every decomposition of the efficiency component necessarily corresponds to a particular grouping of the weighted sum

$$\Xi^{E}=\sum_{i}\sum_{t}\sum_{s^{t}}\omega_{t}^{i}\omega_{t}^{i}\left(s^{t}\right)\frac{dV_{t}^{i|\lambda}\left(s^{t}\right)}{d\theta}.$$

If all individuals value welfare gains over time and across histories equally, then efficiency simply corresponds to the discounted value — using the common discount factor — of aggregate history welfare gains $\sum_i \frac{dV_t^{i|\lambda}(s^t)}{d\theta}$. In general, when individuals have different valuations, Proposition 3 decomposes efficiency gains into an *aggregate efficiency* component, which corresponds to the discounted value of aggregate history welfare gains $\sum_i \frac{dV_t^{i|\lambda}(s^t)}{d\theta}$ using common time and stochastic discount factors, and two components, *intertemporal-sharing* and *risk-sharing,* which respectively capture the differential impact of a perturbation towards individuals with different valuations over dates or histories.[9]

Proposition 3. *(Aggregate Efficiency/Risk-Sharing/Intertemporal-Sharing Decomposition) The efficiency component of a normalized welfare assessment can be decomposed into aggregate efficiency, risk-sharing, and intertemporal-sharing components, Ξ^{AE}, Ξ^{RS}, and Ξ^{IS}, as follows:*

$$\underbrace{\Xi^{E}}_{Efficiency}=\underbrace{\sum_{t}\omega_{t}\sum_{s^{t}}\omega_{t}\left(s^{t}\right)\sum_{i}\frac{dV_{t}^{i|\lambda}\left(s^{t}\right)}{d\theta}}_{\Xi^{AE}\ (Aggregate\ Efficiency)}+\underbrace{\sum_{t}\omega_{t}\sum_{s^{t}}\omega_{t}\left(s^{t}\right)\mathbb{C}ov_{i}^{\Sigma}\left[\frac{\omega_{t}^{i}\left(s^{t}\right)}{\omega_{t}\left(s^{t}\right)},\frac{dV_{t}^{i|\lambda}\left(s^{t}\right)}{d\theta}\right]}_{\Xi^{RS}\ (Risk\text{-}Sharing)}$$
$$+\underbrace{\sum_{t}\omega_{t}\mathbb{C}ov_{i}^{\Sigma}\left[\frac{\omega_{t}^{i}}{\omega_{t}},\frac{dV_{t}^{i|\lambda}}{d\theta}\right]}_{\Xi^{IS}\ (Intertemporal\text{-}Sharing)}, \tag{11}$$

where the averages of normalized weights $\omega_t = \frac{1}{I}\sum_i \omega_t^i$ and $\omega_t(s^t) = \frac{1}{I}\sum_i \omega_t^i(s^t)$ define aggregate time and stochastic discount factors, and where $\mathbb{C}ov_i^{\Sigma}[\cdot,\cdot] = I\cdot\mathbb{C}ov_i^{\Sigma}[\cdot,\cdot]$ denotes a cross-sectional covariance-sum among individuals.

The justification for this decomposition is once again based on the compensation principle, now applied over dates and histories. The sum of normalized date welfare gains at date t, $\sum_i \frac{dV_t^{i|\lambda}}{d\theta}$, corresponds to the aggregate willingness-to-pay for the impact of the perturbation at that date in units of the date welfare numeraire (a unit of date-t consumption). Hence, when $\sum_i \frac{dV_t^{i|\lambda}}{d\theta} > 0$, the winners of the perturbation at date t could hypothetically compensate the losers in terms of the date welfare numeraire at that date. The aggregate time discount factor that makes it possible to add up

[9] By choosing the label risk-sharing and coining the (less conventional) label intertemporal-sharing, we seek to highlight the cross-sectional nature of both components. Terms such as insurance, consumption smoothing, or intertemporal smoothing do not necessarily have a cross-sectional connotation since they can be applied to a single individual.

aggregate gains across different dates, by expressing them in units of the lifetime welfare numeraire, is $\omega_t = \frac{1}{I}\sum_i \omega_t^i$.

Therefore, the *unique* way to decompose Ξ^E into a component that corresponds to the discounted sum — using an aggregate discount factor — of the aggregate willingness-to-pay for the perturbation at each date and its complement is

$$\Xi^E = \sum_i \sum_t \omega_t^i \frac{dV_t^{i|\lambda}}{d\theta} = \underbrace{\sum_t \omega_t \sum_i \frac{dV_t^{i|\lambda}}{d\theta}}_{\Xi^{AE}+\Xi^{RS}} + \underbrace{\sum_t \omega_t \mathbb{C}ov_i^{\Sigma}\left[\frac{\omega_t^i}{\omega_t}, \frac{dV_t^{i|\lambda}}{d\theta}\right]}_{\Xi^{IS}}. \quad (12)$$

The intertemporal-sharing component — which can equivalently be expressed as $\Xi^{IS} = \sum_t \omega_t \sum_i \left(\frac{\omega_t^i}{\omega_t} - 1\right) \frac{dV_t^{i|\lambda}}{d\theta}$ — captures the contribution to efficiency due to differences in valuation over time across individuals. The date-t element of Ξ^{IS} is positive when a perturbation relatively favors individuals with a higher relative valuation (dynamic weight) for date t, and vice versa.

The same logic applies to decomposing aggregate normalized date welfare gains at date t,

$$\sum_i \frac{dV_t^{i|\lambda}}{d\theta} = \sum_i \sum_{s^t} \omega_t^i\left(s^t\right) \frac{dV_t^{i|\lambda}\left(s^t\right)}{d\theta} = \sum_{s^t} \omega_t\left(s^t\right) \sum_i \frac{dV_t^{i|\lambda}\left(s^t\right)}{d\theta} + \sum_{s^t} \omega_t\left(s^t\right) \mathbb{C}ov_i^{\Sigma}\left[\frac{\omega_t^i\left(s^t\right)}{\omega_t\left(s^t\right)}, \frac{dV_t^{i|\lambda}\left(s^t\right)}{d\theta}\right].$$

The sum of normalized history welfare gains at history s^t, $\sum_i \frac{dV_t^{i|\lambda}(s^t)}{d\theta}$, corresponds to the aggregate willingness-to-pay for the impact of the perturbation at that history in units of the history welfare numeraire (a unit of history-s^t consumption). Hence, when $\sum_i \frac{dV_t^{i|\lambda}(s^t)}{d\theta} > 0$, the winners of the perturbation at history s^t could hypothetically compensate the losers in terms of the history welfare numeraire at that history. The aggregate stochastic discount factor that makes it possible to add up aggregate gains across different histories at a given date, by expressing them in units of the date welfare numeraire, is $\omega_t\left(s^t\right) = \frac{1}{I}\sum_i \omega_t^i\left(s^t\right)$.

Therefore, Proposition 3 is the *unique* decomposition in which the efficiency component can be expressed as the discounted sum — using aggregate time and stochastic discount factors — of aggregate history welfare gains, Ξ^{AE}, and its complement, $\Xi^{RS} + \Xi^{IS}$.[10] The risk-sharing component — which can equivalently be expressed as $\Xi^{RS} = \sum_t \omega_t \sum_{s^t} \omega_t\left(s^t\right) \sum_i \left(\frac{\omega_t^i(s^t)}{\omega_t(s^t)} - 1\right) \frac{dV_t^{i|\lambda}(s^t)}{d\theta}$ — captures the contribution to efficiency due to differences in valuation over histories across individuals. The history-s^t element of Ξ^{RS} is positive when a perturbation relatively favors individuals with a higher relative valuation (stochastic weight) for history s^t, and vice versa. In Section H of the Online Appendix, we discuss alternative subdecompositions for the risk-sharing and intertemporal-sharing components.

By construction, the aggregate efficiency component is exclusively a function of aggregate

[10] We adopt the label aggregate efficiency because Ξ^{AE} is the subcomponent of efficiency that uses "aggregate" time and stochastic discount factors to discount "aggregate" history welfare gains.

history welfare gains, while the risk-sharing and intertemporal-sharing components depend on how history welfare gains accrue to individuals with different valuations for particular dates or histories. Importantly, for Ξ^{AE} to be non-zero, it must be that a perturbation changes aggregate history welfare gains at particular dates and histories — see Proposition 4c) below. In Section H of the Appendix, we show how it is possible to subdecompose aggregate efficiency into a component that captures improved smoothing of aggregate history welfare gains — this is the single force behind the cost-of-business-cycles computation in Lucas (1987) — and a component that captures changes in expected aggregate history welfare gains.

3.3.1 Properties of Aggregate Efficiency/Risk-Sharing/Intertemporal-Sharing Decomposition

The differences in normalized dynamic and stochastic weights that govern the risk-sharing and intertemporal-sharing components depend on the extent to which individuals can freely smooth consumption across dates and histories. In Proposition 4, we show that i) risk-sharing and intertemporal-sharing are zero when individual marginal rates of substitution across all dates and histories are equalized (which occurs when markets are complete) and ii) intertemporal-sharing is zero when marginal rates of substitution across dates are equalized across agents (which occurs when all individuals frictionlessly borrow and save).[11]

Proposition 4. *(Properties of Aggregate Efficiency/Risk-Sharing/Intertemporal-Sharing Decomposition)*

a) *(Complete markets) When marginal rates of substitution across all dates and histories are equalized across individuals — a condition that complete markets economies satisfy — the risk-sharing and intertemporal-sharing components are zero:* $\Xi^{RS} = \Xi^{IS} = 0$.

b) *(Frictionless borrowing and saving) When marginal rates of substitution across dates are equalized across individuals — a condition satisfied when all individuals frictionlessly borrow and save — the intertemporal-sharing component is zero:* $\Xi^{IS} = 0$.

c) *(Zero aggregate normalized history welfare gains) When a perturbation features zero aggregate normalized history welfare gains at all dates and histories, the aggregate efficiency component is zero:* $\Xi^{AE} = 0$.

Proposition 4a) and 4b) follow from the fact that complete markets ensure that individual valuations across dates and histories are identical, while frictionless borrowing and saving do the same exclusively

[11] It follows from Proposition 2 that each of Ξ^{AE}, Ξ^{RS}, and Ξ^{IS} are identical for all welfarist planners and invariant to preference-preserving utility transformations.

across dates. Intuitively, normalized dynamic and stochastic weights can be expressed in terms of state-prices as follows:

$$\omega_t^i\left(s^t\right)=\frac{q_t^i\left(s^t\right)}{\sum_{s^t} q_t^i\left(s^t\right)} \qquad \text{and} \qquad \omega_t^i=\frac{\sum_{s^t} q_t^i\left(s^t\right)}{\sum_t \sum_{s^t} q_t^i\left(s^t\right)}, \tag{13}$$

where $q_t^i\left(s^t\right)=\left(\beta^i\right)^t \pi_t\left(s^t\right) \frac{\partial u^i\left(s^t\right)}{\partial c_t^i} / \frac{\partial u^i\left(s^0\right)}{\partial c_0^i}$ denotes individual i's (shadow) date-0 state-price over history s^t. When markets are complete, all valuations are equalized, so $q_t^i=q_t$, $\forall i$, which implies that $\omega_t^i=\omega_t$ and $\omega_t^i\left(s^t\right)=\omega_t\left(s^t\right)$. Hence, in this case, welfare assessments are exclusively driven by aggregate efficiency and redistribution. Under frictionless borrowing and saving, the valuation of zero-coupon bonds is equalized, so $\sum_{s^t} q_t^i\left(s^t\right)=\sum_{s^t} q_t\left(s^t\right)$, which implies that $\omega_t^i=\omega_t$. Proposition 4 also implies that the good/factor on which financial claims are written (e.g., the consumption good or, more generally, a nominal numeraire) is a natural history welfare numeraire. Proposition 4c) shows that the aggregate efficiency component can only take non-zero values for perturbations in which aggregate normalized history welfare gains are non-zero.

Finally, even though — as explained above — the efficiency component is strictly positive for every Pareto-improving perturbation, one or two of the three components of the decomposition of the efficiency component introduced in Proposition 3 may be negative. That is, Pareto efficiency exclusively requires that the sum of the aggregate efficiency, risk-sharing, and intertemporal-sharing components is positive, so $\Xi^E=\Xi^{AE}+\Xi^{RS}+\Xi^{IS}>0$, but not each of them. Example 5 below and Application 2 in Section 4 illustrate this possibility in two different scenarios.

3.3.2 Illustration of Aggregate Efficiency/Risk-Sharing/Intertemporal-Sharing Decomposition

Here, we present three minimal examples that illustrate the Aggregate Efficiency/Risk-Sharing/Intertemporal-Sharing Decomposition. Example 3 illustrates the difference between Ξ^{AE} and Ξ^{IS}, while Example 4 illustrates the difference between Ξ^{AE} and Ξ^{RS}. Example 5 shows how it is possible to find Pareto-improving perturbations in perfect foresight economies in which $\Xi^{AE}<0$, but $\Xi^{IS}>0$. All three examples are designed to feature $\Xi^{RD}=0$, regardless of the choice of social welfare function.

Example 3. [Aggregate Efficiency vs. Intertemporal-Sharing]. This example contrasts two different economies. First, consider a two-date perfect foresight economy with two individuals ($I=2$) with preferences $V^i=u\left(c_0^i\right)+u\left(c_1^i\right)$. Consider a perturbation that smoothes individual consumption across dates by smoothing aggregate consumption, as follows:

$$c_0^i=3-\theta \quad \text{and} \quad c_1^i=1+\theta,\ \forall i.$$

In this case, the welfare gains from varying θ are exclusively due to aggregate efficiency. Aggregate consumption in this example is $2 \times (3-\theta)$ at date 0 and $2 \times (1+\theta)$ at date 1, so the perturbation changes aggregate date welfare gains at both dates. Formally, $\Xi^E = \Xi^{AE} > 0$, while $\Xi^{RS} = \Xi^{IS} = 0$.

Second, consider a different two-date perfect foresight economy with two individuals (I $= 2$) with preferences $V^i = u\left(c_0^i\right) + u\left(c_1^i\right)$. Consider instead a perturbation that smoothes individual consumption across dates by reshuffling consumption within each date but without changing aggregate consumption, as follows:

$$
\begin{aligned}
c_0^1 &= 3-\theta & \qquad c_0^2 &= 1+\theta \\
c_1^1 &= 1+\theta & \qquad c_1^2 &= 3-\theta.
\end{aligned}
$$

In this case, the welfare gains from varying θ are exclusively due to intertemporal-sharing. Aggregate consumption in this example is 4 at both dates 0 and 1, so the perturbation does not change aggregate date welfare gains. Formally, $\Xi^E = \Xi^{IS} > 0$, while $\Xi^{AE} = \Xi^{RS} = 0$.

In the first economy, the aggregate willingness to pay for the perturbation (compensation principle) is negative at date 0 and positive at date 1, and the discounted value, using a common discount factor, makes $\Xi^{AE} > 0$. Since individuals have the same valuation for consumption at both dates, $\Xi^{IS} = 0$. In the second economy, the aggregate willingness to pay for the perturbation (compensation principle) is zero at both dates, so $\Xi^{AE} = 0$. Since individuals have different valuation for consumption at both dates, and the perturbation transfers resources from those who value consumption less to more at each date, $\Xi^{IS} > 0$. This is precisely what the decomposition is designed to achieve.

Example 4. [Aggregate Efficiency vs. Risk-Sharing]. This example contrasts two different economies with risk. First, consider a single-date economy with two equally probable states $s \in \{1, 2\}$ and two individuals ($I = 2$) with preferences $V^i = \sum_s \pi(s)\, u\left(c^i(s)\right)$. Consider a perturbation that smoothes individual consumption across dates by smoothing aggregate consumption, as follows:

$$
c^i(1) = 3-\theta \quad \text{and} \quad c^i(2) = 1+\theta,\ \forall i.
$$

In this case, the welfare gains from varying θ are exclusively due to aggregate efficiency. Aggregate consumption in this example is $2 \times (3-\theta)$ in state 1 and $2 \times (1+\theta)$ in state 2, so the perturbation changes aggregate history welfare gains in both states. Formally, $\Xi^E = \Xi^{AE} > 0$, while $\Xi^{RS} = \Xi^{IS} = 0$.

Second, consider a different single-date economy with two equally probable states $s \in \{1, 2\}$ and two individuals (I $= 2$) with preferences $V^i = \sum_s \pi(s)\, u\left(c^i(s)\right)$. Consider instead a perturbation that smoothes individual consumption across states by reshuffling consumption within each state but without changing aggregate consumption, as follows:

$$c^1(1) = 3 - \theta \qquad c^2(1) = 1 + \theta$$
$$c^1(2) = 1 + \theta \qquad c^2(2) = 3 - \theta.$$

In this case, the welfare gains from varying θ are exclusively due to risk-sharing. Aggregate consumption in this example is 4 in both states, so the perturbation does not change aggregate history welfare gains. Formally, $\Xi^E = \Xi^{RS} > 0$, while $\Xi^{AE} = \Xi^{IS} = 0$.

In the first economy, the aggregate willingness to pay for the perturbation (compensation principle) is negative in state 1 and positive in state 2, and its risk-adjusted value, using a common stochastic discount factor, makes $\Xi^{AE} > 0$. Since individuals have the same valuation for consumption in both states, $\Xi^{RS} = 0$. In the second economy, the aggregate willingness to pay for the perturbation (compensation principle) is zero at both states, so $\Xi^{AE} = 0$. Since individuals have different valuation for consumption at both states, and the perturbation transfers resources from those who value consumption less to more at each state, $\Xi^{RS} > 0$. This is precisely what the decomposition is designed to achieve.

Example 5. [Pareto Improvement with $\Xi^E > 0$, $\Xi^{AE} < 0$, and $\Xi^{IS} > 0$]. At last, we consider a different perturbation to the two-individual two-date perfect foresight economy from Example 3. This alternative perturbation preserves the smoothing of consumption across individuals but also features a permanent aggregate consumption loss, modulated by a parameter $\alpha \geq 0$. This loss could represent, for instance, aggregate technology regress. Formally, individual consumption is given by

$$c_0^1 = 3 - \theta - \alpha\theta \qquad c_0^2 = 1 + \theta - \alpha\theta$$
$$c_1^1 = 1 + \theta - \alpha\theta \qquad c_1^2 = 3 - \theta - \alpha\theta.$$

In this case, the logic from Example 3 applies unchanged, so $\Xi^{IS} > 0$. However, whenever $\alpha > 0$, aggregate consumption falls at both dates, and consequently $\Xi^{AE} < 0$. It should be evident that whenever α is sufficiently small, a marginal increase in θ starting from $\theta = 0$ is a Pareto-improving perturbation that features $\Xi^E = \Xi^{AE} + \Xi^{IS} > 0$ with $\Xi^{AE} < 0$ and $\Xi^{IS} > 0$. This example illustrates that the efficiency component may be positive for perturbations in which aggregate consumption goes down at all dates, provided that the gains from reallocating consumption from individuals with low to high normalized dynamic weights are sufficiently large.

3.4 Additional Properties

In the remainder of this section, we present additional properties of the welfare decomposition introduced in Propositions 1 and 3.[12] Proposition 5 identifies conditions on i) normalized weights and ii) welfare gains that respectively guarantee that Ξ^{RS}, Ξ^{IS}, or Ξ^{RD} are zero.

Proposition 5. *(Properties of Welfare Decomposition: Individual-Invariant Weights or Welfare Gains)*

a) (Individual-invariant normalized weights) If normalized stochastic weights are constant across individuals at all dates and histories, then $\Xi^{RS} = 0$. If normalized dynamic weights are constant across individuals at all dates, then $\Xi^{IS} = 0$. If normalized individual weights are constant across individuals, then $\Xi^{RD} = 0$.

b) (Individual-invariant welfare gains) If history welfare gains $\frac{dV_t^{i|\lambda}(s^t)}{d\theta}$ are identical across individuals at all histories, then $\Xi^{RS} = 0$. If risk-adjusted history welfare gains $\frac{dV_t^{i|\lambda}}{d\theta}$ are identical across individual at all dates, then $\Xi^{IS} = 0$. If lifetime welfare gains are identical across individuals, then $\Xi^{RD} = 0$.

Proposition 5a) shows that invariance of normalized weights across specific dimensions — individual, dynamic, stochastic — implies that redistribution, intertemporal-sharing, and risk-sharing components are respectively zero. This result highlights the cross-sectional nature of these three components, in contrast to aggregate efficiency. Proposition 5b) shows that particular components of the welfare decomposition are zero when perturbations impact all individuals identically at each history, date, or on a lifetime basis.

Proposition 6 identifies which components of the welfare decomposition are zero in particular economies of practical relevance. This result further justifies the labels for the different components of the decomposition and is useful to quickly determine which components of the decomposition are inactive in specific applications.

Proposition 6. *(Properties of Welfare Decomposition: Particular Economies)*

a) (Single individual economies) In single individual ($I = 1$) economies: $\Xi^{RS} = \Xi^{IS} = \Xi^{RD} = 0$.

b) (Perfect foresight economies) In perfect foresight economies: $\Xi^{RS} = 0$.

c) (Economies with ex-ante identical individuals) In economies with ex-ante (but not necessarily ex-post) identical individuals: $\Xi^{IS} = \Xi^{RD} = 0$.

[12] Lemma 1 already implies that normalized individual, dynamic, and stochastic weights and normalized history welfare gains are sufficient statistics to make normalized welfare assessments. These are also the inputs of the different components of the welfare decomposition, making their computation conceptually straightforward.

d) (Static economies) In static ($T = 0$) economies: $\Xi^{RS} = \Xi^{IS} = 0$.

e) (Single good endowment economies) In single good endowment economies in which the aggregate endowment is invariant to the perturbation: $\Xi^{AE} = 0$.

Even though the welfare decomposition is based on the compensation principle, which is formulated in terms of hypothetical transfers between winners and losers, no transfers of resources need to take place for the decomposition to be valid. The decomposition is simply a valuation exercise. That said, in economies in which planners can costlessly and optimally implement transfers of resources among individuals along particular dimensions, Proposition 7 characterizes which components of the welfare decomposition are zero.

Proposition 7. *(Properties of Welfare Decomposition: Transfers)*

a) (Lifetime transfers) If a planner can costlessly and optimally transfer the lifetime welfare numeraire across individuals, then $\Xi^{RD} = 0$.

b) (Date transfers) If a planner can costlessly and optimally transfer the date welfare numeraire across individuals at all dates, then $\Xi^{IS} = \Xi^{RD} = 0$.

c) (History transfers) If a planner can costlessly and optimally transfer the history welfare numeraire across individuals at all histories, then $\Xi^{RS} = \Xi^{IS} = \Xi^{RD} = 0$.

We conclude this section with three remarks.

Remark 1. *(Connection to policy instruments)* Proposition 7 highlights the connection between the different components of the welfare decomposition and how specific policy instruments may come into play to generate welfare gains. For instance, perturbations that feature positive redistribution components imply that policies that transfer the lifetime welfare numeraire across individuals can generate welfare gains. Similarly, perturbations with positive intertemporal-sharing and risk-sharing components imply that policies that transfer date and history welfare numeraires across individuals can also generate welfare gains. The applications studied in Section 4 illustrate this logic further.

Remark 2. *(Welfare decomposition does not rely on optimality)* It is worth highlighting that the welfare decomposition does not rely on individual optimality (i.e., the envelope theorem). The decomposition is exclusively a function of preferences and the considered perturbation. In specific applications, exploiting individual optimality conditions along with budget and resource constraints can simplify the characterization of the components of the decomposition, as the applications in Section 4 illustrate.

Remark 3. *(Term structure)* Welfare assessments as well as each of the components of the welfare decomposition have a term structure of the form

$$\frac{dW^{\lambda}}{d\theta} = \sum_{t} \omega_t \frac{dW_t^{\lambda}}{d\theta} \quad \text{where} \quad \frac{dW_t^{\lambda}}{d\theta} = \Xi_t^{AE} + \Xi_t^{RS} + \Xi_t^{IS} + \Xi_t^{RD}, \tag{14}$$

as we illustrate in Application 1 in Section 4. This structure can be used to compute transition and steady-state welfare gains, and can be refined to define a stochastic structure of welfare gains, as we explain in Section H.1 of the Online Appendix.

4 Applications

In this section, we illustrate how the welfare decomposition introduced in this paper can be used to draw normative conclusions in three scenarios of practical relevance. Table 1 illustrates which components of the welfare decomposition are non-zero in each application.

Table 1: Summary of Applications

#	Application	Ξ^{AE}	Ξ^{RS}	Ξ^{IS}	Ξ^{RD}
1	Consumption Smoothing	$=0$	✓	✓	✓
2	Labor Income Taxation (deterministic)	✓	$=0$	$=0$	✓
2	Labor Income Taxation (stochastic)	✓	✓	$=0$	$=0$
3	Credit Constraint Relaxation	✓	✓	✓	✓

Note: This table illustrates which components of the welfare decomposition are non-zero in each of the applications.

4.1 Application 1: Consumption Smoothing

This application analyzes the welfare effects of a transfer policy that perfectly smooths consumption across individuals who face idiosyncratic consumption risk. The central takeaway is that the persistence of the endowment process determines whether the welfare gains from the transfer policy are attributed to risk-sharing, intertemporal-sharing, or redistribution.

4.1.1 Environment

We consider an infinite-horizon economy with two individuals, $i \in \{1, 2\}$, with identical preferences. We formulate preferences recursively as

$$V^i(s) = u\left(c^i(s)\right) + \beta \sum_{s'} \pi(s'|s) V^i(s'), \quad \text{where} \quad u(c) = \frac{c^{1-\gamma}}{1-\gamma},$$

where $V^i(s)$ and $c^i(s)$ respectively denote the lifetime utility and the consumption of individual i in a given state s; s and s' denote possible states, and $\pi(s'|s)$ denotes Markov transition probabilities; β is a discount factor, and $u(c)$ denotes the instantaneous utility function.

There is a single nonstorable consumption good. We consider an extreme form of incomplete markets: no financial markets. So individual consumption in state s is given by

$$c^i(s) = y^i(s) + \theta T^i(s), \tag{15}$$

where $y^i(s)$ denotes individual i's endowment of the good and $T^i(s)$ denotes the transfer policy, scaled by a parameter $\theta \in [0,1]$. Uncertainty follows a two-state Markov chain. We denote states by $s = \{L, H\}$, standing for low (L) and high (H) realizations of individual 1's endowments, $y^1(s)$. The transition matrix is given by

$$\Pi = \begin{pmatrix} \rho & 1-\rho \\ 1-\rho & \rho \end{pmatrix},$$

where $\rho \in [0,1]$ captures the persistence of the endowment process. To ensure risk is idiosyncratic, we assume that $y^1(s) = \overline{y} + \varepsilon(s)$ and $y^2(s) = \overline{y} - \varepsilon(s)$, where $\overline{y} > 0$, and where $\varepsilon(L) = -\varepsilon(H)$. We consider the welfare assessment of a transfer policy that fully smooths consumption. By setting $T^1(s) = -\varepsilon(s)$ and $T^2(s) = \varepsilon(s)$, individual consumption takes the form

$$c^1(s) = \overline{y} + \varepsilon(s)(1-\theta) \quad \text{and} \quad c^2(s) = \overline{y} - \varepsilon(s)(1-\theta).$$

By varying θ between 0 and 1, this economy transitions from autarky to perfect consumption smoothing. We consider an equal-weighted utilitarian social welfare function, so $\mathcal{W}(V^1, V^2) = V^1 + V^2$, and adopt unit-consumption-based welfare numeraires. Our benchmark parameterization assumes $\beta = 0.95$, $\overline{y} = 1$, $\varepsilon(H) = 0.25$, $\varepsilon(L) = -0.25$, $\gamma = 2$, and $\rho = 0.975$.

4.1.2 Results

Normalized Weights. Figure 2 shows normalized dynamic and stochastic weights — defined in Lemma 1 — when $\theta = 0.25$. Several insights emerge.

First, individuals with an initially low endowment (and high marginal utility) value welfare gains in early periods relatively more than in later periods. And since dynamic weights add up to 1 over time, dynamic weights for different individuals necessarily intersect. Second, stochastic weights show time-dependence despite the stationarity of the model because shocks are persistent. The persistence of the endowment process explains why individuals value early welfare gains more, although the impact of the initial state eventually dissipates. In the long run, individuals value welfare gains more (less) in states with low (high) consumption, as expected. Finally, a normalized

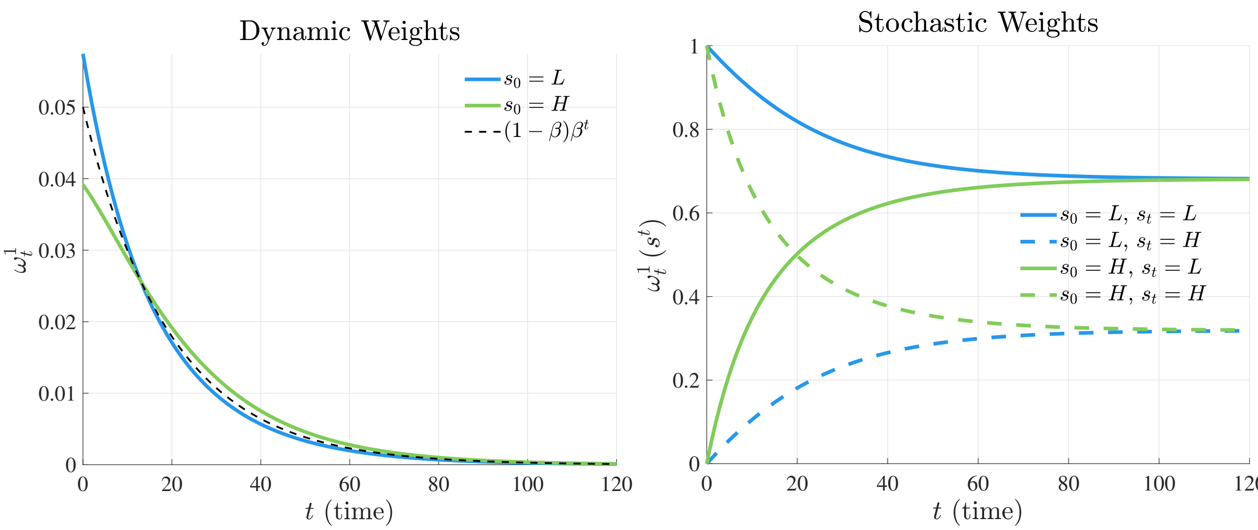

Figure 2: Normalized Weights (Application 1)

Note: This figure shows normalized weights for individual 1 when $\theta = 0.25$. The left panel shows the dynamic weight, ω_t^i, as a function of time when for different initial states $s_0 = \{H, L\}$. For reference, it also shows the dynamic weight for a risk-neutral individual, given by $(1-\beta)\beta^t = \beta^t / \sum_t \beta^t$. The right panel shows the stochastic weights $\omega_t^i(s^t)$ as a function of time for different initial and final states, $s_0 = \{H, L\}$ and $s_t = \{H, L\}$. The individual weights are $\omega^1(s_0 = L) = 1.186$ and $\omega^1(s_0 = H) = 0.814$. Since the model is symmetric, normalized weights for individual 2 can be read off the weights for individual 1 switching the initial state.

utilitarian planner values lifetime welfare gains for the low endowment individual at the time of the assessment at roughly 46% more than for the high endowment individual, since

$$\frac{\omega^1}{\omega^2} = \frac{1.186}{0.814} \approx 1.46, \quad \text{when} \quad s_0 = L.$$

Welfare Decomposition. Figure 3 shows the welfare decomposition for three different parameterizations: $\rho = \{0.5, 0.975, 0.999\}$ when $s_0 = L$; welfare assessments are identical when $s_0 = H$.[13]

This application illustrates how the persistence of endowment shocks changes the relative importance of each of the components of the welfare decomposition. When shocks are transitory ($\rho = 0.5$), risk-sharing dominates, with intertemporal-sharing playing a smaller role and redistribution being virtually zero. When shocks are persistent ($\rho = 0.975$), welfare gains are partly attributed to redistribution, which is larger than intertemporal-sharing, although risk-sharing is still the dominant component. When shocks are virtually permanent ($\rho = 0.999$), redistribution dominates, with risk-sharing and intertemporal-sharing playing a much smaller role. This application is constructed so that the normalized welfare assessment $\frac{dW_t^\lambda}{d\theta}$ is invariant to the level of persistence ρ, underscoring

[13] If the model featured a state in which individuals are identical, the welfare decomposition at that state would be significantly different. Proposition 5 implies that $\Xi^{IS} = \Xi^{RD} = 0$ in that case, so every welfare assessment would exclusively be due to risk-sharing. This fact underscores that the decomposition of welfare assessments critically depends on the state in which an assessment takes place.

shifts in the relative contribution of each of the components of the decomposition. Despite the overall invariance of the welfare assessments, it is important to highlight that knowing the contribution of each component is relevant to determine the precise underlying frictions and which specific policy instruments may generate welfare gains, as described in Remark 1.

We make four additional observations. First, since this is a single good endowment economy in which transfers cancel out in the aggregate, Proposition 6 implies that $\Xi^{AE} = 0$. Second, the optimal policy for a utilitarian planner features perfect consumption smoothing ($\theta^{\star} = 1$). This application is constructed so that the three non-zero components (risk-sharing, intertemporal-sharing, and redistribution) independently conclude that perfect smoothing is optimal. Hence, in practice, the rationale justifying such a policy can significantly differ depending on primitives. Third, as shown in the bottom right panel of Figure 3, intertemporal-sharing is hump-shaped, peaking at $\rho = 0.96$. Intuitively, the difference in valuations induced by the inability to borrow and save is maximal when shocks are persistent. Finally, even though a utilitarian planner finds perfect smoothing optimal, $\theta = 1$ is not a Pareto improvement relative to $\theta = 0$: the individual with a higher initial endowment at the time of the assessment is worse off for values of θ near 1, more so when shocks are more persistent.

Term Structure. Figure 4 shows the term structure of welfare assessments, based on equation (14). The normalized date-t welfare assessment, $\frac{dW_t^{\lambda}}{d\theta}$, is only slightly front-loaded. However, the time-invariance of $\frac{dW_t^{\lambda}}{d\theta}$ masks significant variation in each of its components. The risk-sharing component, which is zero at $t = 0$ and positive at all times, captures all long-run gains from the policy. This occurs because the smoothing policy has risk-sharing benefits at all dates, since $\mathbb{C}ov_i^{\Sigma}\left[\frac{\omega_t^i(s^t)}{\omega_t(s^t)}, \frac{dV_t^{i|\lambda}(s^t)}{d\theta}\right] > 0$ at all later times after $t = 0$.

In contrast, both the intertemporal-sharing and redistribution components are positive at $t = 0$ but end up contributing negatively to the welfare assessment of the policy. Since the normalized date welfare gains $\frac{dV_t^{i|\lambda}}{d\theta}$ converge to the same (positive) value for both individuals when $t \to \infty$, then $\lim_{t\to\infty} \Xi_t^{IS} = 0$. The date on which the dynamic weights of both individuals intersect determines when Ξ_t^{IS} turns negative. The fact that $\lim_{t\to\infty} \Xi_t^{RD} < 0$ is due to the fact that the individual with the initially low endowment (and high marginal utility) values long-run welfare gains relatively less. The subtle behavior of Ξ_t^{IS} and Ξ_t^{RD} is due to the fact that the dynamic weights intersect, as explained in detail in Section D of the Online Appendix.

4.2 Application 2: Labor Income Taxation

This application contrasts the welfare effects of (linear) labor income taxes in two environments: i) a deterministic environment in which individuals differ in their productivity at the time of the welfare assessment, and ii) a stochastic environment in which individuals are identical at the time of the

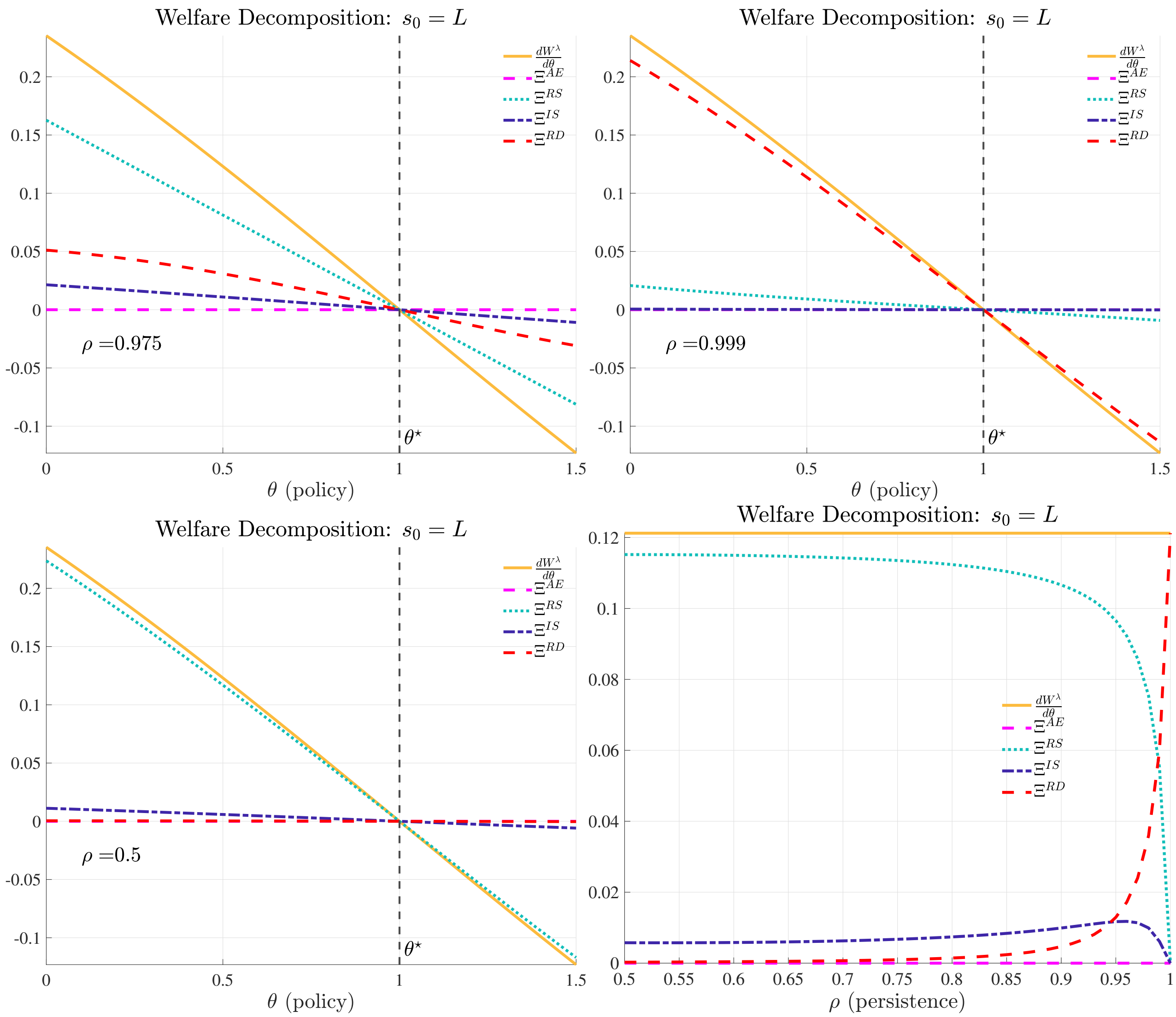

Figure 3: Welfare Decomposition (Application 1)

Note: This figure shows the welfare assessment and its components as a function of the perturbation parameter θ when $s_0 = L$ for three parameterizations: $\rho = 0.975$ (top panel; benchmark), $\rho = 0.5$ (bottom left panel), and $\rho = 0.999$ (top right panel), when $s_0 = L$. The bottom right panel shows the welfare gains from the smoothing policy (integrating marginal welfare gains between $\theta = 0$ and $\theta = 1$) as a function of the persistence parameter. This figure illustrates that the persistence of the endowment process determines whether the welfare gains from the transfer policy are attributed to risk-sharing, intertemporal-sharing, and redistribution

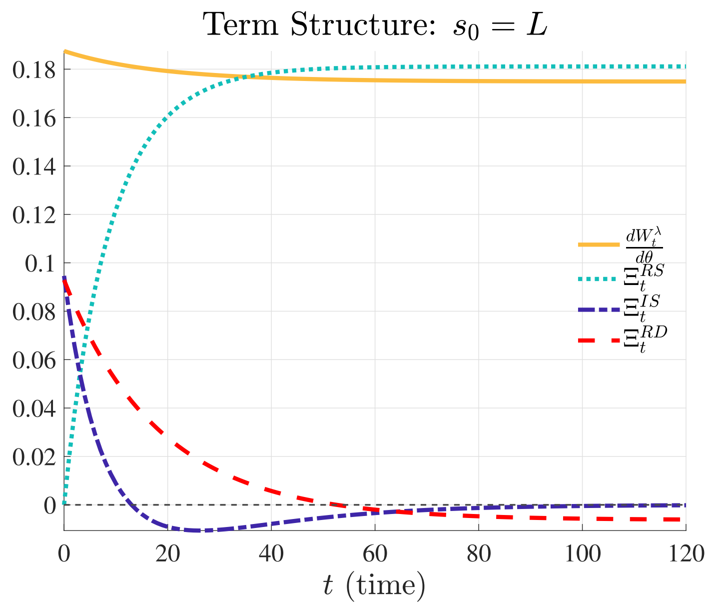

Figure 4: Term Structure of Welfare Decomposition (Application 1)

Note: This figure shows the term structure of welfare assessments, $\frac{dW_t^\lambda}{d\theta}$, and its non-zero components: Ξ_t^{RS}, Ξ_t^{IS}, and Ξ_t^{RD}, as defined in equation (14) and in Section D of the Online Appendix, when $s_0 = L$.

welfare assessment, but experience different shocks. While both environments can be parameterized to yield a quantitatively identical optimal tax, a utilitarian planner attributes the welfare gains from the tax to redistribution in the deterministic environment and to risk-sharing in the stochastic environment. Moreover, in the stochastic environment *all* welfarist planners agree on the magnitude of the optimal tax, while in the deterministic environment the optimal tax is sensitive to the choice of welfare function.

4.2.1 Deterministic Earnings

Environment. We first consider a single-date environment with two individuals $i \in \{1, 2\}$ who make a consumption-labor decision subject to a linear tax in labor earnings. Formally, individuals have identical preferences given by

$$V^i = u\left(c^i, n^i\right),$$

where c^i denotes consumption and n^i hours worked. Individual budget constraints are given by

$$c^i = (1 - \tau) w^i n^i + g,$$

where τ is the linear tax rate and g is a uniform per-capita grant (demogrant) that must satisfy $g = \frac{1}{I} \tau \sum_i w^i n^i$, and w^i denotes individual i's wage. We consider an equal-weighted utilitarian social welfare function, so $\mathcal{W}\left(V^1, V^2\right) = V^1 + V^2$, and adopt unit-consumption-based welfare numeraires.

To simplify the exposition, we assume that preferences take the form $u(c,n) = \frac{1}{1-\gamma}\left(c - \alpha\frac{n^{\sigma}}{\sigma}\right)^{1-\gamma}$. Our parameterization assumes $w^1 = 1$, $w^2 = 5$, $\gamma = 0.5$, $\sigma = 2$, and $\alpha = 1$.

Welfare Decomposition. We now consider the welfare effects of changing the linear tax rate τ where the demogrant g adjusts to satisfy the government's budget constraint. The lifetime welfare gains for individual i induced by a marginal tax change are given by

$$\frac{dV^{i|\lambda}}{d\tau} = \frac{\frac{dV^i}{d\tau}}{\lambda^i} = -w^i n^i + \frac{dg}{d\tau}, \tag{16}$$

where $\frac{dg}{d\tau} = \frac{1}{I}\left(\sum_i w^i n^i + \tau \sum_i w^i \frac{dn^i}{d\tau}\right)$. Since we are considering a static framework, Proposition 6 implies that $\Xi^{RS} = \Xi^{IS} = 0$, so the welfare decomposition exclusively features aggregate efficiency and redistribution. In fact, the welfare assessment of a change in the tax rate can be decomposed as follows:

$$\frac{dW^{\lambda}}{d\tau} = \underbrace{-\tau \sum_i w^i \left(-\frac{dn^i}{d\tau}\right)}_{\Xi^{AE}\text{ (Aggregate Efficiency)}} + \underbrace{\mathbb{C}ov_i^{\Sigma}\left[\omega^i, -w^i n^i\right]}_{\Xi^{RD}\text{ (Redistribution)}}, \quad \text{where} \quad \omega^i = \frac{\frac{\partial u(c^i,n^i)}{\partial c}}{\frac{1}{I}\sum_i \frac{\partial u(c^i,n^i)}{\partial c}}. \tag{17}$$

As illustrated in the left panel of Figure 5, aggregate efficiency welfare gains are 0 at $\tau = 0$ and become increasingly negative as τ increases. These losses capture how the tax reduces the desire to work by individuals. Redistribution gains are strictly positive but decreasing, so this optimal taxation problem is well-behaved and features an optimal interior tax $\tau^{\star}$ that optimally trades off aggregate efficiency losses with redistribution gains.[14]

4.2.2 Random Earnings

Environment. We now consider an environment in which two identical individuals $i \in \{1, 2\}$ face uninsured earnings risk. At the time of the welfare assessment, individuals have expected utility of the form

$$V^i = \sum_s \pi(s)\, u\left(c^i(s), n^i(s)\right),$$

where $c^i(s)$ and $n^i(s)$ denote consumption and hours work in state s. For simplicity, we assume that there are two possible states $s = \{H, L\}$, with probability $\pi(s) = \frac{1}{2}$. To ensure that risk is idiosyncratic, we assume that, in state $s = H$, wages are given by $w^1(H) = \overline{w}$ and $w^2(H) = \underline{w}$, while in state L, $w^1(L) = \underline{w}$ and $w^2(H) = \overline{w}$, where $\overline{w} > \underline{w}$. After the state is realized, individuals make a consumption-labor decision facing a linear tax in labor earnings, and face budget constraints

[14] The optimal linear income tax problem is first studied by Sheshinski (1972). See Piketty and Saez (2013) and Kaplow (2022) for recent surveys of this area.

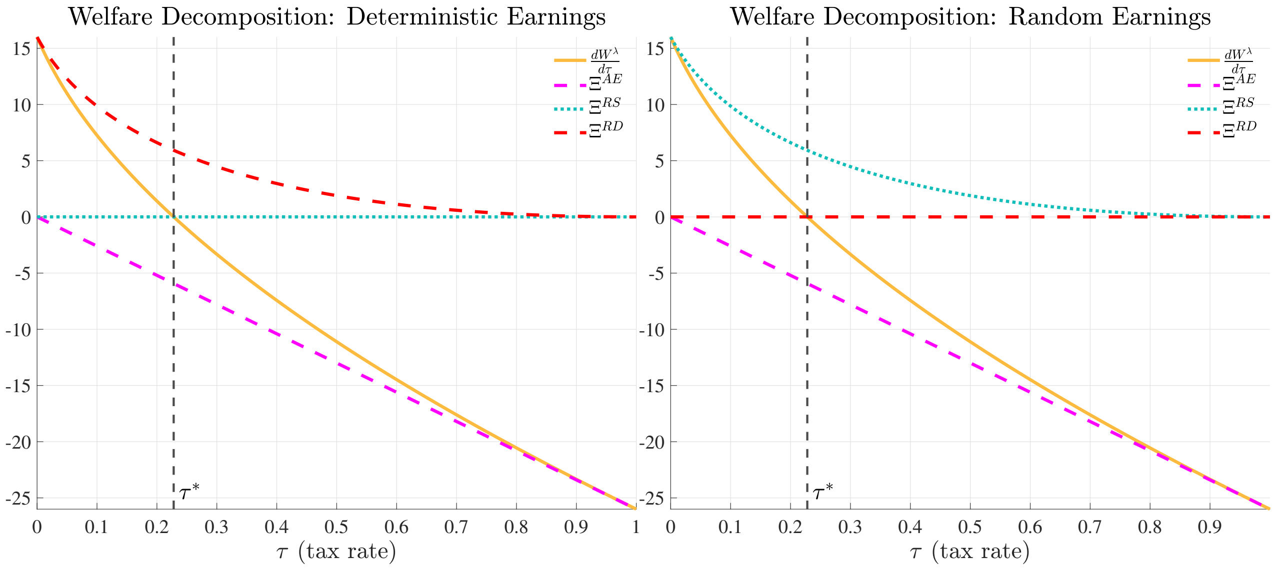

Figure 5: Welfare Decomposition (Application 2)

Note: This figure shows the welfare assessment and the components of the welfare decomposition as a function of the tax rate τ for both the deterministic and random earnings models.

given by

$$c^i(s) = (1-\tau) w^i(s) n^i(s) + g,$$

where τ is the linear tax rate, which we assume to be state-independent, and $g = \frac{1}{I}\tau \sum_i w^i(s) n^i(s)$ is a demogrant, which given our assumptions is also state-independent — for this reason, we write g rather than $g(s)$. We again consider an equal-weighted utilitarian social welfare function, so $\mathcal{W}(V^1, V^2) = V^1 + V^2$, and adopt unit-consumption-based welfare numeraires. We assume again that preferences take the form $u(c^i, n^i) = \frac{1}{1-\gamma}\left(c - \alpha \frac{n^\sigma}{\sigma}\right)^{1-\gamma}$. Our parameterization assumes $\underline{w} = 1$, $\overline{w} = 5$, $\gamma = 0.5$, $\sigma = 2$, and $\alpha = 1$.

Welfare Decomposition. We again consider the welfare effects of changing the linear tax rate τ where the demogrant g adjusts to satisfy the government's budget constraint at each state. The lifetime welfare gains for individual i induced by a marginal tax change are given by

$$\frac{dV^{i|\lambda}}{d\tau} = \frac{\frac{dV^i}{d\tau}}{\lambda^i} = \sum_s \omega_1^i(s)\left[-w^i(s) n^i(s) + \frac{dg}{d\tau}\right] \quad \text{where} \quad \omega_1^i(s) = \frac{\pi(s)\frac{\partial u(c^i(s), n^i(s))}{\partial c(s)}}{\sum_s \pi(s)\frac{\partial u(c^i(s), n^i(s))}{\partial c(s)}}, \tag{18}$$

and where $\frac{dg}{d\tau} = \frac{1}{I}\left(\sum_i w^i(s) n^i(s) + \tau \sum_i w^i(s) \frac{dn^i(s)}{d\tau}\right)$. Since we are considering a framework with ex-ante identical individuals, Proposition 6 implies that $\Xi^{IS} = \Xi^{RD} = 0$, so the welfare decomposition exclusively features aggregate efficiency and risk-sharing. In fact, the welfare assessment of a change in the tax rate can be decomposed as follows:

$$\frac{dW^{\lambda}}{d\tau} = \underbrace{-\tau \sum_{s} \omega_{1}(s) \sum_{i} w^{i}(s) \left(-\frac{dn^{i}(s)}{d\tau}\right)}_{\Xi^{AE} \text{ (Aggregate Efficiency)}} + \underbrace{\sum_{s} \omega_{1}(s) \mathbb{C}ov_{i}^{\Sigma}\left[\frac{\omega_{1}^{i}(s)}{\omega_{1}(s)}, -w^{i}(s) n^{i}(s)\right]}_{\Xi^{RS} \text{ (Risk-Sharing)}}, \tag{19}$$

where $\omega(s) = \frac{1}{I}\sum_{i}\omega^{i}(s)$ and where we use the fact that the normalized individual weight ω^{i} is identical across individuals. As illustrated in the right panel of Figure 5, aggregate efficiency welfare losses are 0 at $\tau = 0$ and become increasingly negative as τ increases, for the same reason as in the deterministic case. Risk-sharing gains are strictly positive but decreasing, as in the deterministic case. Hence, this optimal taxation problem is also well-behaved and features an optimal interior tax $\tau^{\star}$ that optimally trades off aggregate efficiency losses with risk-sharing gains.

Remark 4. *(Pareto Improvement with $\Xi^{E} > 0$, $\Xi^{AE} < 0$, and $\Xi^{RS} > 0$)* A tax increase in the random earnings model illustrates how a welfare assessment can concurrently feature $\Xi^{E} > 0$ and $\Xi^{AE} < 0$. In this economy, an increase in the tax rate below $\tau^{\star}$ is indeed a Pareto improvement, which necessarily implies that $\Xi^{E} > 0$. In that region, aggregate consumption and, more importantly, aggregate history welfare gains decrease as τ increases, which implies that $\Xi^{AE} < 0$. However, the gains from reallocating consumption from individuals with low to high normalized stochastic weights are sufficiently large to make such tax increases desirable and ultimately a positive tax optimal.

4.2.3 Comparison of Deterministic and Random Earnings Models

This application shows that the welfare decomposition can be used to formalize that the deterministic and random earnings models feature different equity-efficiency tradeoffs. While the literature on labor income taxation has conceptually made this distinction — see Piketty and Saez (2013) and Kaplow (2022) — our decomposition provides a formal systematic procedure to understand the rationales that justify optimal taxes in different environments.

Figure 5 precisely compares these two models. While we have parametrized both models so as to yield the same normalized welfare assessment $\frac{dW^{\lambda}}{d\tau}$ and optimal tax $\tau^{\star}$, the welfare decomposition shows that the rationale justifying the optimal tax in both models is substantially different. Drawing this distinction is important because it points out to which policy instruments would make a linear income tax unnecessary. In particular, in the deterministic earnings model, a planner would need access to ex-ante transfers across individuals. In the random earnings model, the planner would need access to transfers across individuals contingent on possible state.

In both models, increasing the tax rate has identical distortionary effects reducing labor supply, which leads to a reduction in aggregate history welfare gains and a negative aggregate efficiency component. However, in the deterministic model the source of welfare gains is redistribution across individuals, while in the random earnings model the source of welfare gains is risk-sharing. Moreover, in the random earnings model *all* welfarist planners agree on the magnitude of the optimal tax, while in the deterministic model the optimal tax is sensitive to the choice of welfare function. This occurs because in the random earnings model there is no equity-efficiency tradeoff: all welfare gains are efficiency gains. In more realistic models in which individuals are both heterogeneous at the assessment and face uninsured risks (see e.g. Heathcote, Storesletten and Violante (2017)), both Ξ^{AE}, Ξ^{RS}, and Ξ^{RD} (and typically Ξ^{IS} in a model with more dates) will interact non-trivially to shape the optimal policy.

4.3 Application 3: Credit Constraint Relaxation

This application studies the welfare implications of a change in credit conditions in an economy in which borrowing-constrained individuals make an investment decision. Varying the borrowing limit in this economy is a tractable perturbation that parameterizes changes in the degree of market completeness. This application, in which all four components of the welfare decomposition are non-zero, illustrates how the welfare decomposition is useful to uncover subtle normative implications of a perturbation.

4.3.1 Environment

We consider a two-date economy populated by two (types of) individuals, $i \in \{1, 2\}$, with identical preferences, given by

$$u\left(c_0^i\right) + \beta \sum_s \pi\left(s\right) u\left(c_1^i\left(s\right)\right),$$

where c_0^i and $c_1^i\left(s\right)$ denote consumption of the single consumption good, $\pi\left(s\right)$ denotes the probabilities of different states at date 1; β is a discount factor, and $u\left(c\right)$ denotes the instantaneous utility function. Since this application assumes perfect competition, individuals in this model correspond to a continuum of agents in equal measure. We refer to $i = 1$ individuals as investors and to $i = 2$ individuals as lenders.

Investors face budget constraints given by

$$c_0^i = n_0 + q_0 b_1^i - \Upsilon^i\left(k_0^i\right)$$
$$c_1^i\left(s\right) = n_1\left(s\right) + z\left(s\right) k_0^i - b_1^i,$$

where n_0 and $n_1\left(s\right)$ denote endowments of the consumption good, b_1^i denotes the face value of the amount borrowed at price q_0^i (the interest rate in this economy is $1/q_0^i$), and $\Upsilon^i\left(k_0^i\right)$ denotes the cost of producing k_0^i units of capital at date 0, which yields $z\left(s\right)$ units at date 1 in state s. For simplicity, we assume that there are two states $s = \{H, L\}$, with $z\left(H\right) > z\left(L\right)$, and that $\sum_s \pi\left(s\right) z\left(s\right)$ is

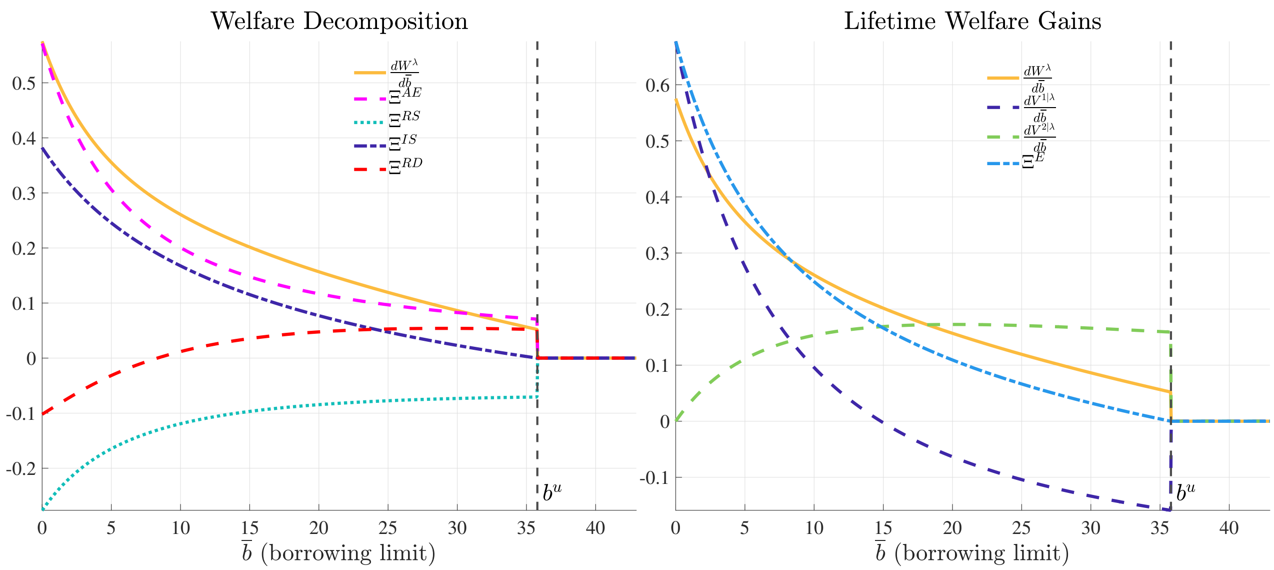

Figure 6: Welfare Decomposition and Lifetime Welfare Gains (Application 3)

Note: The left panel of this figure shows the welfare assessment and the components of the welfare decomposition as a function of the borrowing limit $\bar{b}$. The right panel shows the normalized lifetime welfare gains for investors and lenders, the normalized welfare assessment, and the efficiency component, as a function of the borrowing limit $\bar{b}$.

sufficiently large so that investors always find it optimal to invest and borrow. Lenders face identical budget constraints, but cannot operate the capital technology, so $\Upsilon^2\left(k_0^2\right) = k_0^2 = 0$.

Investors can borrow up to a predetermined borrowing limit $\bar{b}$:

$$b_1^i \leq \bar{b}.$$

Therefore, this economy features two forms of market incompleteness: i) investors cannot arrange insurance from lenders against the investment risk they bear since they only have access to a non-contingent security, and ii) investors and lenders cannot frictionlessly borrow and save when the borrowing constraint binds.[15]

An equilibrium is characterized by allocations $\left\{c_0^i, c_1^i\left(s\right), b_1^i, k_0^1\right\}$ and a price of the riskless asset q_0 that clears the market for borrowing and saving, so that $\sum_i b_1^i = 0$. When solving the model, we assume that $\Upsilon^1\left(k_0^1\right) = \frac{\phi}{2}\left(k_0^1\right)^2$ and $u\left(c\right) = \frac{c^{1-\gamma}}{1-\gamma}$. Our parameterization assumes $\beta = 0.95$, $\gamma = 1.5$, $\phi = 0.1$, $z\left(L\right) = 5$, $z\left(H\right) = 35$, $\pi\left(L\right) = 0.7$, and $n_0^i = n_1^i\left(s\right) = 40$.

[15] An alternative exercise that we do not explore here is to understand the welfare impact of changing the aggregate amount of public debt — see, for instance, Woodford (1990), or more recently Azzimonti and Yared (2019), among others.

4.3.2 Welfare Decomposition

We consider the welfare effects of varying the borrowing limit $\bar{b}$ from 0, which corresponds to an autarky economy, until b^u, the level at which the borrowing constraint ceases to bind. The left panel in Figure 6 shows the normalized welfare assessment and the welfare decomposition associated with this perturbation. The right panel shows normalized lifetime welfare gains for investors and lenders, the normalized welfare assessment, and the efficiency component.

Figure 6 illustrates how changes in the borrowing limit impact welfare through the four components of the welfare decomposition, with each component taking the following sign:

$$\frac{dW^\lambda}{d\bar{b}} = \underbrace{\Xi^{AE}}_{>0} + \underbrace{\Xi^{RS}}_{<0} + \underbrace{\Xi^{IS}}_{>0} + \underbrace{\Xi^{RD}}_{\gtrless 0}.$$

First, as we relax the borrowing limit, investors are able to invest more, which increases discounted (using an aggregate discount factor) aggregate history welfare gains, which in turn implies $\Xi^{AE} > 0$. Since relaxing the borrowing limit also increases the resources available to investors at date 0, when their relative valuation is higher, this implies $\Xi^{IS} > 0$. As the borrowing limit approaches the unconstrained level of borrowing b^u, intertemporal-sharing tends towards zero because, at that point, dynamic weights are equalized across investors and lenders. In contrast, Ξ^{AE} is still strictly positive. This is explained by the fact that markets remain incomplete, so the economy is not at a Pareto optimal allocation.

Second, as we relax the borrowing limit and investors increase their investment, their consumption becomes relatively more exposed to the investment risk, which they are unable to share with lenders: formally $\mathbb{C}ov_i^{\Sigma}\left[\frac{\omega_1^i(s)}{\omega_1(s)}, \frac{dc_1^i(s)}{d\bar{b}}\right] < 0$, where $\frac{dc_1^i(s)}{d\bar{b}} = z(s)\frac{dk_0^i}{d\bar{b}} - \frac{db_1^i}{d\bar{b}}$. This justifies why $\Xi^{RS} < 0$ and illustrates how making markets more complete — in the sense of relaxing a borrowing constraint — may be associated with a negative risk-sharing component, a phenomenon that may seem counterintuitive at first. Similar to Ξ^{AE}, as the borrowing limit approaches b^u, Ξ^{RS} is still strictly negative, as markets remain incomplete for any level of $\bar{b}$.

Third, as we relax the borrowing limit, Ξ^{RD} switches from negative to positive. In this economy, the value of a unit of perpetual consumption for lenders is higher than for investors since the former lack access to the profitable investment technology. This difference explains why the normalized individual weights of a utilitarian planner are $\omega^1 \approx 0.82$ and $\omega^2 \approx 1.18$, favoring welfare gains by lenders. As we show in the Appendix, exploiting optimality conditions, it is possible to show that lifetime welfare gains take the form

$$\frac{dV^i}{d\bar{b}} = \underbrace{\left(u'\left(c_0^i\right)q_0 - \beta \sum_s \pi\left(s\right)u'\left(c_1^i\left(s\right)\right)\right)\frac{db_1^i}{d\bar{b}}}_{\text{Direct Borrowing Effect}} + \underbrace{u'\left(c_0^i\right)\frac{dq_0}{d\bar{b}}b_1^i}_{\substack{\text{Distributive}\\\text{Pecuniary Effect}}}. \tag{20}$$

The direct borrowing effect in (20) is zero for lenders and strictly positive for constrained investors, while the distributive pecuniary effects are zero-sum in units of date-0 consumption, that is $\sum_i \frac{dq_0}{d\bar{b}} b_1^i = 0$ (Dávila and Korinek, 2018). Because relaxing the borrowing limit increases borrowing and interest rates, $\frac{dq_0}{d\bar{b}} < 0$, the distributive pecuniary effect hurts those who borrow (investors) and benefits those who lend (lenders). The right panel in Figure 6 shows that relaxing the borrowing limit for low values of $\bar{b}$ benefits both investors and lenders, with the former benefiting more. As the borrowing limit $\bar{b}$ increases, investors' marginal welfare gains are reduced and eventually turn negative. This occurs because each individual investor fails to internalize how borrowing more increases interest rates in the competitive equilibrium, hurting other investors. The right panel of Figure 6 shows that Ξ^{RD} turns negative when the normalized lifetime welfare gains for investors and lenders intersect.

Finally, it is worth highlighting that the efficiency component as a whole, $\Xi^E = \Xi^{AE} + \Xi^{RS} + \Xi^{IS}$, is always positive, becoming zero as the borrowing limit approaches b^u. While it is natural that the efficiency implications of relaxing the constraint approach zero as the constraint ceases to bind, this result implies that $\Xi^{AE} + \Xi^{RS} = 0$ when $\bar{b} = b^u$, further explaining why aggregate efficiency and risk-sharing take opposite signs at that point.

5 Additional Results and Extensions

The Online Appendix presents several extensions and additional results, which we summarize here.

Generalized Welfare Criteria: DS-planners Section E leverages the welfare decomposition to systematically construct non-welfarist welfare criteria based on individual, dynamic, and stochastic weights. This approach allows us to formalize normative objectives that isolate specific components of the welfare decomposition. Because this normative approach entails defining weights for each time, history, and individual as a primitive of the welfare assessment, we say it is based on Dynamic Stochastic Generalized Social Marginal Welfare Weights (dynamic–stochastic weights or DS-weights, for short). These newly introduced DS-planners have the potential to allow for disciplined discussions about the mandates of independent technocratic institutions (central banks, financial regulators, other regulatory agencies, etc.).

Generalized Environments. Section F derives the welfare decomposition for general welfare numeraires and discusses the implications of different numeraire choices. Section G describes how to extend our results to more general environments. First, we show how to use the welfare decomposition in environments with heterogeneous beliefs, both for welfarist and non-welfarist planners. Second, we describe how to allow for recursive preferences, in particular, the widely used Epstein-Zin preferences. We also consider the case of non-time separable non-expected utility preferences. Third, we show that allowing for multiple consumption goods and factors simply requires redefining history welfare gains. Fourth, we describe how to consider perturbations that entail changes in probabilities. Fifth, we show how to generalize the welfare decomposition to scenarios in which normalized individual, dynamic, or stochastic weights are zero. Finally, we briefly discuss how to implement the decomposition in environments with idiosyncratic and aggregate states, a continuum of individuals, dates, or histories, and non-differentiabilities.

Subdecompositions and Alternative Decompositions. Section H describes how to further decompose the components of the welfare decomposition. In addition to the term structure decomposition already described, we show that each individual can be attributed a particular share of each of the components of the welfare decomposition. Next, we show that it is possible to decompose each of the components into a term due to consumption or factor supply growth and a term due to reallocation. We also show how to construct a stochastic decomposition of aggregate efficiency into expected aggregate efficiency and aggregate smoothing, and of redistribution into expected redistribution and redistributive smoothing. Finally, we provide two alternative cross-sectional decompositions of the risk-sharing and intertemporal-sharing components.

Additional Results. Section I includes additional results. First, we study properties of the welfare decomposition for allocations that solve the Pareto problem. Second, we explain how the ability to costlessly transfer resources across individuals by a planner impacts the welfare decomposition by limiting cross-sectional variation in normalized weights. Third, we explain how to translate marginal welfare assessments into global welfare assessments. Finally, we provide bounds based on the dispersion of normalized weights and welfare gains for Ξ^{RS}, Ξ^{IS}, and Ξ^{RD}.

6 Conclusion

This paper introduces a decomposition of welfare assessments for general dynamic stochastic economies with heterogeneous individuals. Guided by the compensation principle, it initially decomposes a welfare assessment into an efficiency and a redistribution component, while the efficiency component is further decomposed into i) aggregate efficiency, ii) intertemporal-sharing, and iii) risk-sharing components. The decomposition is based on constructing individual, dynamic, and stochastic weights that characterize how welfarist planners make tradeoffs across individuals, dates, and histories.

Retrospectively, the welfare decomposition opens the door to revisiting the exact rationales that have justified welfarist welfare assessments in existing work. Looking forward, we hope our results inform ongoing and future discussions on the desirability of particular policies, the welfare impact of shocks, and the design of policy-making mandates.

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Appendix

A Proofs and Derivations: Section 3

Proof of Lemma 1. (Normalized Welfare Gains and Normalized Weights)

Proof. We can express an (unnormalized) welfare assessment $\frac{dW}{d\theta}$ as

$$\frac{dW}{d\theta} = \sum_i \frac{\partial \mathcal{W}}{\partial V^i} \frac{dV^i}{d\theta} = \sum_i \frac{\partial \mathcal{W}}{\partial V^i} \lambda^i \frac{\frac{dV^i}{d\theta}}{\lambda^i},$$

where our choice of lifetime welfare numeraire is such that $\lambda^i = \sum_t \left(\beta^i\right)^t \sum_{s^t} \pi_t\left(s^t\right) \frac{\partial u_t^i\left(s^t\right)}{\partial c_t^i}$. Hence, the normalized welfare assessment takes the form

$$\frac{dW^\lambda}{d\theta} = \frac{\frac{dW}{d\theta}}{\frac{1}{I}\sum_i \frac{\partial \mathcal{W}}{\partial V^i}\lambda^i} = \sum_i \omega^i \frac{\frac{dV^i}{d\theta}}{\lambda^i}, \quad \text{where} \quad \omega^i = \frac{\frac{\partial \mathcal{W}}{\partial V^i}\lambda^i}{\frac{1}{I}\sum_i \frac{\partial \mathcal{W}}{\partial V^i}\lambda^i}.$$

We can then express individual i's lifetime welfare gains in units of the lifetime welfare numeraire as

$$\begin{aligned}
\frac{dV^{i|\lambda}}{d\theta} = \frac{\frac{dV^i}{d\theta}}{\lambda^i} &= \sum_t \sum_{s^t} \frac{\left(\beta^i\right)^t \pi_t\left(s^t\right)}{\lambda^i} \frac{dV_t^i\left(s^t\right)}{d\theta} = \sum_t \sum_{s^t} \frac{\left(\beta^i\right)^t \pi_t\left(s^t\right) \frac{\partial u_t^i\left(s^t\right)}{\partial c_t^i}}{\lambda^i} \frac{\frac{dV_t^i\left(s^t\right)}{d\theta}}{\frac{\partial u_t^i\left(s^t\right)}{\partial c_t^i}} \\
&= \sum_t \frac{\sum_{s^t}\left(\beta^i\right)^t \pi_t\left(s^t\right) \frac{\partial u_t^i\left(s^t\right)}{\partial c_t^i}}{\lambda^i} \sum_{s^t} \frac{\left(\beta^i\right)^t \pi_t\left(s^t\right) \frac{\partial u_t^i\left(s^t\right)}{\partial c_t^i}}{\sum_{s^t}\left(\beta^i\right)^t \pi_t\left(s^t\right) \frac{\partial u_t^i\left(s^t\right)}{\partial c_t^i}} \frac{dV_t^{i|\lambda}\left(s^t\right)}{d\theta} \\
&= \sum_t \omega_t^i \sum_{s^t} \omega_t^i\left(s^t\right) \frac{dV_t^{i|\lambda}\left(s^t\right)}{d\theta},
\end{aligned}$$

where $\omega_t^i = \frac{\sum_{s^t}\left(\beta^i\right)^t \pi_t\left(s^t\right)\lambda_t^i\left(s^t\right)}{\sum_t \left(\beta^i\right)^t \sum_{s^t} \pi_t\left(s^t\right) \frac{\partial u_t^i\left(s^t\right)}{\partial c_t^i}}$, $\omega_t^i\left(s^t\right) = \frac{\left(\beta^i\right)^t \pi_t\left(s^t\right)\lambda_t^i\left(s^t\right)}{\sum_{s^t}\left(\beta^i\right)^t \pi_t\left(s^t\right)\lambda_t^i\left(s^t\right)}$, and $\frac{dV_t^{i|\lambda}\left(s^t\right)}{d\theta} = \frac{\frac{dV_t^i\left(s^t\right)}{d\theta}}{\frac{\partial u_t^i\left(s^t\right)}{\partial c_t^i}}$ is given in equation (6). □

Proof of Proposition 1. (Efficiency/Redistribution Decomposition)

Proof. For any two random variables x_i and y_i, it follows that $\sum_i x_i y_i = \frac{1}{I}\sum_i x_i \sum_i y_i + \mathbb{C}ov_i^\Sigma\left[x_i, y_i\right]$, where $\mathbb{C}ov_i^\Sigma\left[x_i, y_i\right] = I \cdot \mathbb{C}ov_i\left[x_i, y_i\right]$. Equation (10) follows from

$$\frac{dW^\lambda}{d\theta} = \sum_i \omega^i \frac{dV^{i|\lambda}}{d\theta} = \underbrace{\sum_i \frac{dV^{i|\lambda}}{d\theta}}_{\Xi^E} + \underbrace{\mathbb{C}ov_i^\Sigma\left[\omega^i, \frac{dV^{i|\lambda}}{d\theta}\right]}_{\Xi^{RD}},$$

where we use the fact that $\frac{1}{I}\sum_i \omega^i = 1$. This is the unique decomposition of the weighted sum $\sum_i \omega^i \frac{dV^{i|\lambda}}{d\theta}$ into an unweighted sum and its complement. □

Proof of Proposition 2. (Properties of Efficiency/Redistribution Decomposition)

Proof. a) This result follows from the fact that the social welfare function exclusively impacts the definition of ω^i, and that $\frac{dV^{i|\lambda}}{d\theta}$ — equivalently, normalized dynamic and stochastic weights — is invariant to the social welfare function.

b) This result follows from the fact that $\frac{dV^{i|\lambda}}{d\theta}$ — equivalently, normalized dynamic and stochastic weights — is invariant to the preference-preserving transformations considered.

c) This result follows immediately from the definitions of Ξ^{RD} and ω^i. □

Proof of Proposition 3. (Aggregate Efficiency/Risk-Sharing/Intertemporal-Sharing Decomposition)

Proof. Starting from the definition of the efficiency component:

$$\Xi^E = \sum_i \frac{\frac{dV^i}{d\theta}}{\lambda^i} = \sum_i \sum_t \omega_t^i \sum_{s^t} \omega_t^i \left(s^t\right) \frac{dV_t^{i|\lambda}\left(s^t\right)}{d\theta} = \sum_i \sum_t \omega_t^i \frac{dV_t^{i|\lambda}}{d\theta} = \sum_t \sum_i \omega_t^i \frac{dV_t^{i|\lambda}}{d\theta}$$
$$= \sum_t \omega_t \left(\sum_i \frac{dV_t^{i|\lambda}}{d\theta} + \mathbb{C}ov_i^{\Sigma}\left[\frac{\omega_t^i}{\omega_t}, \frac{dV_t^{i|\lambda}}{d\theta}\right]\right) = \sum_t \omega_t \sum_i \frac{dV_t^{i|\lambda}}{d\theta} + \underbrace{\sum_t \omega_t \mathbb{C}ov_i^{\Sigma}\left[\frac{\omega_t^i}{\omega_t}, \frac{dV_t^{i|\lambda}}{d\theta}\right]}_{\Xi^{IS}},$$

where $\omega_t = \frac{1}{I}\sum_i \omega_t^i$ and $\frac{dV_t^{i|\lambda}}{d\theta} = \sum_{s^t} \omega_t^i\left(s^t\right) \frac{dV_t^{i|\lambda}\left(s^t\right)}{d\theta}$. Note that $\mathbb{C}ov_i^{\Sigma}\left[\frac{\omega_t^i}{\omega_t}, \frac{dV_t^{i|\lambda}}{d\theta}\right] = \sum_i \left(\frac{\omega_t^i}{\omega_t} - 1\right) \frac{dV_t^{i|\lambda}}{d\theta}$. This is the unique decomposition of the weighted sum $\sum_i \frac{\omega_t^i}{\omega_t} \frac{dV_t^{i|\lambda}}{d\theta}$ into an unweighted sum and its complement. Moreover,

$$\sum_i \frac{dV_t^{i|\lambda}}{d\theta} = \sum_i \sum_{s^t} \omega_t^i\left(s^t\right) \frac{dV_t^{i|\lambda}\left(s^t\right)}{d\theta} = \sum_{s^t} \sum_i \omega_t^i\left(s^t\right) \frac{dV_t^{i|\lambda}\left(s^t\right)}{d\theta}$$
$$= \sum_{s^t} \omega_t\left(s^t\right) \left(\sum_i \frac{dV_t^{i|\lambda}\left(s^t\right)}{d\theta} + \mathbb{C}ov_i^{\Sigma}\left[\frac{\omega_t^i\left(s^t\right)}{\omega_t\left(s^t\right)}, \frac{dV_t^{i|\lambda}\left(s^t\right)}{d\theta}\right]\right)$$
$$= \underbrace{\sum_{s^t} \omega_t\left(s^t\right) \sum_i \frac{dV_t^{i|\lambda}\left(s^t\right)}{d\theta}}_{\Xi_t^{AE}} + \underbrace{\sum_{s^t} \omega_t\left(s^t\right) \mathbb{C}ov_i^{\Sigma}\left[\frac{\omega_t^i\left(s^t\right)}{\omega_t\left(s^t\right)}, \frac{dV_t^{i|\lambda}\left(s^t\right)}{d\theta}\right]}_{\Xi_t^{RS}},$$

where $\omega_t\left(s^t\right) = \frac{1}{I}\sum_i \omega_t^i\left(s^t\right)$, and where $\Xi^{AE} = \sum_t \omega_t \Xi_t^{AE}$ and $\Xi^{RS} = \sum_t \omega_t \Xi_t^{RS}$. Note that $\mathbb{C}ov_i^{\Sigma}\left[\frac{\omega_t^i(s^t)}{\omega_t(s^t)}, \frac{dV_t^{i|\lambda}(s^t)}{d\theta}\right] = \sum_i \left(\frac{\omega_t^i(s^t)}{\omega_t(s^t)} - 1\right) \frac{dV_t^{i|\lambda}(s^t)}{d\theta}$. This is the unique decomposition of the weighted sum $\sum_i \frac{\omega_t^i(s^t)}{\omega_t(s^t)} \frac{dV_t^{i|\lambda}(s^t)}{d\theta}$ into an unweighted sum and its complement. □

Online Appendix

Section C of this Online Appendix provides detailed derivations for Examples 1 through 5. Section D includes proofs and derivations for Section 4 of the paper. Section E constructs non-welfarist welfare criteria based on individual, dynamic, and stochastic weights. Section F derives the welfare decomposition for general welfare numeraires. Sections G, H, and I include extensions and additional results, and Section J relates our results to existing work.

B Proofs and Derivations: Section 3 (cont.)

Proof of Proposition 4. (Properties of Aggregate Efficiency/Risk-Sharing/Intertemporal-Sharing Decomposition)

Proof. a) When marginal rates of substitution are equalized across all dates and histories across individuals, $\omega_t^i(s^t) = \omega_t(s^t)$ and $\omega_t^i = \omega_t$. Alternatively, from equation (13), when markets are complete there is a unique stochastic discount factor, which implies that $q_t^i(s^t) = q_t(s^t)$. Proposition 5a) then implies that $\Xi^{RS} = \Xi^{IS} = 0$.

b) When marginal rates of substitution are equalized across all dates across individuals, $\omega_t^i = \omega_t$. Alternatively, from equation (13), when individuals frictionlessly borrow and save, $\sum_{s^t} q_t^i(s^t)$ is identical across individuals. Proposition 5a) then implies that $\Xi^{IS} = 0$.

c) This result follows from the definition of Ξ^{AE} when $\sum_i \frac{dV_t^{i|\lambda}(s^t)}{d\theta} = 0$, $\forall s^t$. □

Proof of Proposition 5. (Properties of Welfare Decomposition: Individual-Invariant Weights or Welfare Gains)

Proof. a) If $\omega_t^i(s^t)$ are identical across individuals, $\mathbb{C}ov_i^{\Sigma}\left[\omega_t^i(s^t), \frac{dV_t^{i|\lambda}(s^t)}{d\theta}\right] = 0$, $\forall t$, $\forall s^t$, so $\Xi^{RS} = 0$. If ω_t^i are identical across individuals, $\mathbb{C}ov_i^{\Sigma}\left[\omega_t^i, \frac{dV_t^{i|\lambda}}{d\theta}\right] = 0$, $\forall t$, so $\Xi^{IS} = 0$. If ω^i are identical across individuals, $\mathbb{C}ov_i^{\Sigma}\left[\omega^i, \frac{dV^{i|\lambda}}{d\theta}\right] = 0$, so $\Xi^{RD} = 0$.

b) If $\frac{dV_t^{i|\lambda}(s^t)}{d\theta}$ are identical across individuals, $\mathbb{C}ov_i^{\Sigma}\left[\omega_t^i(s^t), \frac{dV_t^{i|\lambda}(s^t)}{d\theta}\right] = 0$, $\forall t$, $\forall s^t$, so $\Xi^{RS} = 0$. If $\frac{dV_t^{i|\lambda}}{d\theta}$ are identical across individuals, $\mathbb{C}ov_i^{\Sigma}\left[\omega_t^i, \frac{dV_t^{i|\lambda}}{d\theta}\right] = 0$, $\forall t$, so $\Xi^{IS} = 0$. If $\frac{dV^{i|\lambda}}{d\theta}$ are identical across individuals, $\mathbb{C}ov_i^{\Sigma}\left[\omega^i, \frac{dV^{i|\lambda}}{d\theta}\right] = 0$, so $\Xi^{RD} = 0$. □

Proof of Proposition 6. (Properties of Welfare Decomposition: Particular Economies)

Proof. a) If $I = 1$, all normalized weights are trivially identical across individuals. The result then follows from Proposition 5a).

b) If there is no risk, $\omega_t^i(s^t) = 1$, $\forall s^t$. The result then follows from Proposition 5a).

c) If individuals are ex-ante identical, ω^i and ω_t^i are identical across individuals. The result then follows from Proposition 5a).

d) Since we have assumed that $\pi_0\left(s^0|s_0\right)=1$, $\omega_0^i=\omega_0^i\left(s^t\right)=1$ when $T=0$.[16] The result then follows from Proposition 5a).

e) In a single good endowment economy, the single good is the only possible history welfare numeraire. Hence $\sum_i \frac{dV_t^{i|\lambda}\left(s^t\right)}{d\theta}=\sum_i \frac{dc_t^i\left(s^t\right)}{d\theta}=0$, where the last equality follows from the fact that the aggregate consumption equals the aggregate endowment, which is invariant to the perturbation. The result then follows from the definition of Ξ^{AE}. □

Proof of Proposition 7. (Properties of Welfare Decomposition: Transfers)

Proof. In a) a planner sets transfers so that ω^i is identical across individuals. In b), a planner sets transfers so that ω_t^i is identical across individuals. In c), a planner sets transfers so that $\omega_t^i\left(s^t\right)$ is identical across individuals. The results then follow from Proposition 5a). □

C Detailed Derivations: Examples Section 3

Whenever it is useful for illustration, we assume that $u\left(c\right)=\log\left(c\right)$ and that $\theta=0$.

C.1 Example 1

In this simple economy, dynamic and stochastic weights are equal to 1. Note that $\frac{dc^A}{d\theta}=1$ and $\frac{dc^B}{d\theta}=-1$, so $\sum_i \frac{dc^i}{d\theta}=0$.

Individual weights are given by

$$\omega^i=\frac{\alpha^i u'\left(c^i\right)}{\frac{1}{2}\sum_i \alpha^i u'\left(c^i\right)}=\frac{\alpha^i \frac{1}{c^i}}{\frac{1}{2}\sum_i \alpha^i \frac{1}{c^i}}=\frac{\alpha^i}{\frac{1}{2}\sum_i \alpha^i} \quad \text{if } \theta=0,$$

where α^i denotes the Pareto weights.

The welfare decomposition is

$$\frac{dW^\lambda}{d\theta}=\frac{dW}{\frac{1}{2}\sum_i \alpha^i u'\left(c^i\right)}=\sum_i \omega^i \frac{dc^i}{d\theta}=\underbrace{\sum_i \frac{dc^i}{d\theta}}_{\Xi^{AE}}+\underbrace{\mathbb{C}ov_i^\Sigma\left[\omega^i,\frac{dc^i}{d\theta}\right]}_{\Xi^{RD}},$$

[16] Models with a single date but multiple histories — as the random earnings scenario in Application 2 — can be interpreted as a two-date model in which instantaneous utility is zero at the initial date: see Section G.5 of the Online Appendix.

where

$$\Xi^{AE} = \sum_i \frac{dc^i}{d\theta} = 0$$

$$\Xi^{RD} = \sum_i \left(\omega^i - 1\right) \frac{dc^i}{d\theta} = \omega^A - \omega^B = \frac{\alpha^A - \alpha^B}{\frac{1}{2}\left(\alpha^A + \alpha^B\right)} \quad \text{if } \theta = 0.$$

As stated in the text, Ξ^{RD} can take on any sign depending on ω^i. If the individual weight for individual A is higher than for B, $\omega^A > \omega^B$, and $\Xi^{RD} > 0$. If instead $\omega^A < \omega^B$, then $\Xi^{RD} < 0$.

C.2 Example 2

In this simple economy, dynamic and stochastic weights are equal to 1. Note that $\frac{dc^A}{d\theta} = 2$ and $\frac{dc^B}{d\theta} = 1$, so $\sum_i \frac{dc^i}{d\theta} = 3$.

As in Example 1, Individual weights are given by

$$\omega^i = \frac{\alpha^i u'\left(c^i\right)}{\frac{1}{2}\sum_i \alpha^i u'\left(c^i\right)} = \frac{\alpha^i \frac{1}{c^i}}{\frac{1}{2}\sum_i \alpha^i \frac{1}{c^i}} = \frac{\alpha^i}{\frac{1}{2}\sum_i \alpha^i} \quad \text{if } \theta = 0,$$

where α^i denotes the Pareto weights.

The welfare decomposition is

$$\frac{dW^\lambda}{d\theta} = \frac{dW}{\frac{1}{2}\sum_i \alpha^i u'\left(c^i\right)} = \sum_i \omega^i \frac{dc^i}{d\theta} = \underbrace{\sum_i \frac{dc^i}{d\theta}}_{\Xi^{AE}} + \underbrace{\mathbb{C}ov_i^\Sigma\left[\omega^i, \frac{dc^i}{d\theta}\right]}_{\Xi^{RD}},$$

where

$$\Xi^{AE} = \sum_i \frac{dc^i}{d\theta} = 3$$

$$\Xi^{RD} = \sum_i \left(\omega^i - 1\right) \frac{dc^i}{d\theta} - = \left(\omega^A - 1\right) 2 + \left(\omega^B - 1\right) = \frac{\omega^A - \omega^B}{2} = \frac{\alpha^A - \alpha^B}{\alpha^A + \alpha^B} \quad \text{if } \theta = 0.$$

As stated in the text, Ξ^{RD} can take on any sign depending on ω^i. If the individual weight for individual A is higher than for B, $\omega^A > \omega^B$, and $\Xi^{RD} > 0$. If instead $\omega^A < \omega^B$, then $\Xi^{RD} < 0$.

C.3 Example 3

First Economy. Individual and stochastic weights are $\omega^i = \omega_1^i(s) = 1$. Dynamic weights, which are identical for both individuals, are given by

$$\omega_0^i = \frac{u'\left(c_0^i\right)}{u'\left(c_0^i\right)+u'\left(c_1^i\right)} = \frac{\frac{1}{3-\theta}}{\frac{1}{3-\theta}+\frac{1}{1+\theta}} = \frac{1}{4} \quad \text{if} \quad \theta = 0$$
$$\omega_1^i = \frac{u'\left(c_1^i\right)}{u'\left(c_0^i\right)+u'\left(c_1^i\right)} = \frac{\frac{1}{1+\theta}}{\frac{1}{3-\theta}+\frac{1}{1+\theta}} = \frac{3}{4} \quad \text{if} \quad \theta = 0.$$

Hence, $\omega_0 = \frac{1}{4}$ and $\omega_1 = \frac{3}{4}$ if $\theta = 0$. Note that $\frac{dc_0^i}{d\theta} = -1$ and $\frac{dc_1^i}{d\theta} = 1$, so

$$\frac{dc_0}{d\theta} = \sum_i \frac{dc_0^i}{d\theta} = -2 \quad \text{and} \quad \frac{dc_1}{d\theta} = \sum_i \frac{dc_1^i}{d\theta} = 2.$$

The welfare decomposition is

$$\frac{dW^\lambda}{d\theta} = \Xi^E = \Xi^{AE} = \omega_0 \frac{dc_0}{d\theta} + \omega_1 \frac{dc_1}{d\theta} = \frac{1}{4}(-2) + \frac{3}{4}(+2) = 1.$$

Second Economy. Individual and stochastic weights are $\omega^i = \omega_1^i(s) = 1$. Dynamic weights for individual 1 are

$$\omega_0^1 = \frac{u'\left(c_0^1\right)}{u'\left(c_0^1\right)+u'\left(c_1^1\right)} = \frac{\frac{1}{3-\theta}}{\frac{1}{3-\theta}+\frac{1}{1+\theta}} = \frac{1}{4} \quad \text{if} \quad \theta = 0$$
$$\omega_1^1 = \frac{u'\left(c_1^1\right)}{u'\left(c_0^1\right)+u'\left(c_1^1\right)} = \frac{\frac{1}{1+\theta}}{\frac{1}{3-\theta}+\frac{1}{1+\theta}} = \frac{3}{4} \quad \text{if} \quad \theta = 0.$$

Dynamic weights for individual 2 are

$$\omega_0^2 = \frac{u'\left(c_0^2\right)}{u'\left(c_0^2\right)+u'\left(c_1^2\right)} = \frac{\frac{1}{1+\theta}}{\frac{1}{3-\theta}+\frac{1}{1+\theta}} = \frac{3}{4} \quad \text{if} \quad \theta = 0$$
$$\omega_1^2 = \frac{u'\left(c_1^1\right)}{u'\left(c_0^2\right)+u'\left(c_1^2\right)} = \frac{\frac{1}{3-\theta}}{\frac{1}{3-\theta}+\frac{1}{1+\theta}} = \frac{1}{4} \quad \text{if} \quad \theta = 0.$$

Note that

$$\omega_0 = \frac{1}{2}\sum_i \omega_0^i = \frac{1}{2}\left(\frac{1}{4}+\frac{3}{4}\right) = \frac{1}{2} \quad \text{and} \quad \omega_1 = \frac{1}{2}\sum_i \omega_1^i = \frac{1}{2}\left(\frac{3}{4}+\frac{1}{4}\right) = \frac{1}{2} \quad \text{if} \quad \theta = 0.$$

Note that $\frac{dc_0^1}{d\theta} = -1$, $\frac{dc_0^2}{d\theta} = 1$, $\frac{dc_1^1}{d\theta} = 1$, and $\frac{dc_1^2}{d\theta} = -1$, so

$$\frac{dc_0}{d\theta} = \sum_i \frac{dc_0^i}{d\theta} = 0 \quad \text{and} \quad \frac{dc_1}{d\theta} = \sum_i \frac{dc_1^i}{d\theta} = 0.$$

The welfare decomposition is

$$\begin{aligned}\frac{dW^\lambda}{d\theta} = \Xi^E = \Xi^{IS} &= \omega_0 \sum_i \left(\frac{\omega_0^i}{\omega_0} - 1\right) \frac{dc_0^i}{d\theta} + \omega_1 \sum_i \left(\frac{\omega_1^i}{\omega_1} - 1\right) \frac{dc_1^i}{d\theta} = \\ &= \frac{1}{2}\left(\left(\frac{1}{2} - 1\right)(-1) + \left(\frac{3}{2} - 1\right)(+1)\right) + \frac{1}{2}\left(\left(\frac{3}{2} - 1\right)(+1) + \left(\frac{1}{2} - 1\right)(-1)\right) \\ &= 1.\end{aligned}$$

C.4 Example 4

First Economy. Individual and dynamic weights are $\omega^i = \omega_1^i = 1$. Stochastic weights, which are identical for both individuals, are given by

$$\begin{aligned}\omega_1^i(1) &= \frac{\pi(1)\, u'(c^i(1))}{\sum_s \pi(s)\, u'(c^i(s))} = \frac{\frac{1}{3-\theta}}{\frac{1}{1+\theta} + \frac{1}{3-\theta}} = \frac{1}{4} \quad \text{if} \quad \theta = 0 \\ \omega_1^i(2) &= \frac{\pi(2)\, u'(c^i(2))}{\sum_s \pi(s)\, u'(c^i(s))} = \frac{\frac{1}{1+\theta}}{\frac{1}{1+\theta} + \frac{1}{3-\theta}} = \frac{3}{4} \quad \text{if} \quad \theta = 0.\end{aligned}$$

Hence $\omega_1(1) = \frac{1}{4}$ and $\omega_1(2) = \frac{3}{4}$ if $\theta = 0$. Note that $\frac{dc^i(1)}{d\theta} = -1$ and $\frac{dc^i(2)}{d\theta} = 1$, so

$$\frac{dc(1)}{d\theta} = \sum_i \frac{dc^i(1)}{d\theta} = -2 \quad \text{and} \quad \frac{dc(2)}{d\theta} = \sum_i \frac{dc^i(2)}{d\theta} = 2.$$

The welfare decomposition is

$$\frac{dW^\lambda}{d\theta} = \Xi^E = \Xi^{AE} = \omega_1(1)\frac{dc(1)}{d\theta} + \omega_1(2)\frac{dc(2)}{d\theta} = \frac{1}{4}(-2) + \frac{3}{4}(+2) = 1.$$

Second Economy. Individual and dynamic weights are $\omega^i = \omega_1^i = 1$. Stochastic weights for individual 1 are

$$\begin{aligned}\omega_1^1(1) &= \frac{\pi(1)\, u'(c^i(1))}{\sum_s \pi(s)\, u'(c^i(s))} = \frac{\frac{1}{3-\theta}}{\frac{1}{1+\theta} + \frac{1}{3-\theta}} = \frac{1}{4} \quad \text{if} \quad \theta = 0 \\ \omega_1^1(2) &= \frac{\pi(2)\, u'(c^i(2))}{\sum_s \pi(s)\, u'(c^i(s))} = \frac{\frac{1}{1+\theta}}{\frac{1}{1+\theta} + \frac{1}{3-\theta}} = \frac{3}{4} \quad \text{if} \quad \theta = 0.\end{aligned}$$

Stochastic weights for individual 2 are

$$\omega_1^2(1) = \frac{\pi(1) u'(c^2(1))}{\sum_s \pi(s) u'(c^2(s))} = \frac{\frac{1}{1+\theta}}{\frac{1}{1+\theta} + \frac{1}{3-\theta}} = \frac{3}{4} \quad \text{if} \quad \theta = 0$$
$$\omega_1^2(2) = \frac{\pi(2) u'(c^2(2))}{\sum_s \pi(s) u'(c^2(s))} = \frac{\frac{1}{3-\theta}}{\frac{1}{1+\theta} + \frac{1}{3-\theta}} = \frac{1}{4} \quad \text{if} \quad \theta = 0.$$

Note also that

$$\omega_1(1) = \frac{1}{2}\sum_i \omega_1^i(1) = \frac{1}{2} \quad \text{and} \quad \omega_1(2) = \frac{1}{2}\sum_i \omega_1^i(2) = \frac{1}{2} \quad \text{if} \quad \theta = 0,$$

Note that $\frac{dc^1(1)}{d\theta} = -1$, $\frac{dc^1(2)}{d\theta} = 1$, $\frac{dc^1(2)}{d\theta} = 1$, and $\frac{dc^1(2)}{d\theta} = 1$, so

$$\frac{dc(1)}{d\theta} = \sum_i \frac{dc^i(1)}{d\theta} = 0 \quad \text{and} \quad \frac{dc(2)}{d\theta} = \sum_i \frac{dc^i(2)}{d\theta} = 0.$$

The welfare decomposition is

$$\begin{aligned}
\frac{dW^\lambda}{d\theta} = \Xi^E = \Xi^{RS} &= \sum_s \omega_1(s) \sum_i \left(\frac{\omega_1^i(s)}{\omega_1(s)} - 1\right) \frac{dc^i(s)}{d\theta} \\
&= \frac{1}{2}\left(\left(\frac{1}{2} - 1\right)(-1) + \left(\frac{3}{2} - 1\right)(+1) + \left(\frac{1}{2} - 1\right)(-1) + \left(\frac{3}{2} - 1\right)(+1)\right) \\
&= 1.
\end{aligned}$$

C.5 Example 5

Individual and stochastic weights are $\omega^i = \omega_1^i(s) = 1$. Dynamic weights for individual 1 are

$$\omega_0^1 = \frac{u'(c_0^1)}{u'(c_0^1) + u'(c_1^1)} = \frac{\frac{1}{3-\theta-\alpha\theta}}{\frac{1}{3-\theta-\alpha\theta} + \frac{1}{1+\theta-\alpha\theta}} = \frac{1}{4} \quad \text{if } \theta = 0$$
$$\omega_1^1 = \frac{u'(c_0^1)}{u'(c_0^1) + u'(c_1^1)} = \frac{\frac{1}{1+\theta-\alpha\theta}}{\frac{1}{3-\theta-\alpha\theta} + \frac{1}{1+\theta-\alpha\theta}} = \frac{3}{4} \quad \text{if } \theta = 0.$$

Dynamic weights for individual 2 are

$$\omega_0^2 = \frac{u'(c_0^2)}{u'(c_0^2) + u'(c_1^2)} = \frac{\frac{1}{1+\theta-\alpha\theta}}{\frac{1}{3-\theta-\alpha\theta} + \frac{1}{1+\theta-\alpha\theta}} = \frac{3}{4} \quad \text{if } \theta = 0$$
$$\omega_1^2 = \frac{u'(c_0^2)}{u'(c_0^2) + u'(c_1^1)} = \frac{\frac{1}{3-\theta-\alpha\theta}}{\frac{1}{3-\theta-\alpha\theta} + \frac{1}{1+\theta-\alpha\theta}} = \frac{1}{4} \quad \text{if } \theta = 0.$$

Note that

$$\omega_0 = \frac{1}{2}\sum_i \omega_0^i = \frac{1}{2} \quad \text{and} \quad \omega_1 = \frac{1}{2}\sum_i \omega_1^i = \frac{1}{2} \quad \text{if } \theta = 0.$$

Note that $\frac{dc_0^1}{d\theta} = -1-\alpha$, $\frac{dc_0^2}{d\theta} = 1-\alpha$, $\frac{dc_1^1}{d\theta} = 1-\alpha$, and $\frac{dc_1^2}{d\theta} = -1-\alpha$, so

$$\frac{dc_0}{d\theta} = \sum_i \frac{dc_0^i}{d\theta} = -2\alpha \quad \text{and} \quad \frac{dc_1}{d\theta} = \sum_i \frac{dc_1^i}{d\theta} = -2\alpha.$$

The welfare decomposition is

$$\frac{dW^\lambda}{d\theta} = \underbrace{\omega_0 \sum_i \frac{dc_0^i}{d\theta} + \omega_1 \sum_i \frac{dc_1^i}{d\theta}}_{\Xi^{AE}} + \underbrace{\sum_i \left(\frac{\omega_0^i}{\omega_0} - 1\right)\frac{dc_0^i}{d\theta} + \omega_1 \sum_i \left(\frac{\omega_1^i}{\omega_1} - 1\right)\frac{dc_1^i}{d\theta}}_{\Xi^{IS}} = 1 - 2\alpha,$$

where

$$\begin{aligned}
\Xi^{AE} &= \frac{1}{2}(-2\alpha) + \frac{1}{2}(-2\alpha) = -2\alpha \\
\Xi^{IS} &= \frac{1}{2}\left(-\frac{1}{2}(-1-\alpha) + \frac{1}{2}(1-\alpha)\right) + \frac{1}{2}\left(\frac{1}{2}(1-\alpha) - \frac{1}{2}(-1-\alpha)\right) = 1.
\end{aligned}$$

At $\theta = 0$, $\Xi^{AE} < 0$ and $\Xi^{IS} > 0.$, where the overall welfare effect is positive whenever $\alpha < \frac{1}{2}$.

D Detailed Derivations: Applications Section 4

D.1 Application 1

Figure OA-1 here explains the behavior of Ξ_t^{IS} and Ξ_t^{RD} in Figure 4 in the text. Note that

$$\Xi_t^{IS} = \mathbb{C}ov_i^\Sigma\left[\frac{\omega_t^i}{\omega_t}, \frac{dV_t^{i|\lambda}}{d\theta}\right] \quad \text{and} \quad \Xi_t^{RD} = \mathbb{C}ov_i^\Sigma\left[\omega^i, \frac{\omega_t^i}{\omega_t}\frac{dV_t^{i|\lambda}}{d\theta}\right].$$

The left panel in Figure OA-1 shows that the risk-discounted welfare gains for individual i at date t, $\frac{dV_t^{i|\lambda}}{d\theta}$, are initially positive for the individual with the lower endowment at $s_0 = L$ (individual 1) and negative for the individual with the higher endowment, although they both converge to a positive common value. This captures the fact that the policy initially hurts the individual who starts with a high endowment and benefits the individual who starts with a low endowment, but as time goes by, the identity of the favored individual is uncertain. In the long run, both individuals are favored by the policy by eliminating consumption risk.

The right panel in Figure OA-1 shows that the risk-discounted welfare gains for individual i at date t relative to the average, $\frac{\omega_t^i}{\omega_t}\frac{dV_t^{i|\lambda}}{d\theta}$, converge to positive values for both individuals, but initially negative and higher in the long run for individual 2 (with the higher endowment at $s_0 = L$). Because

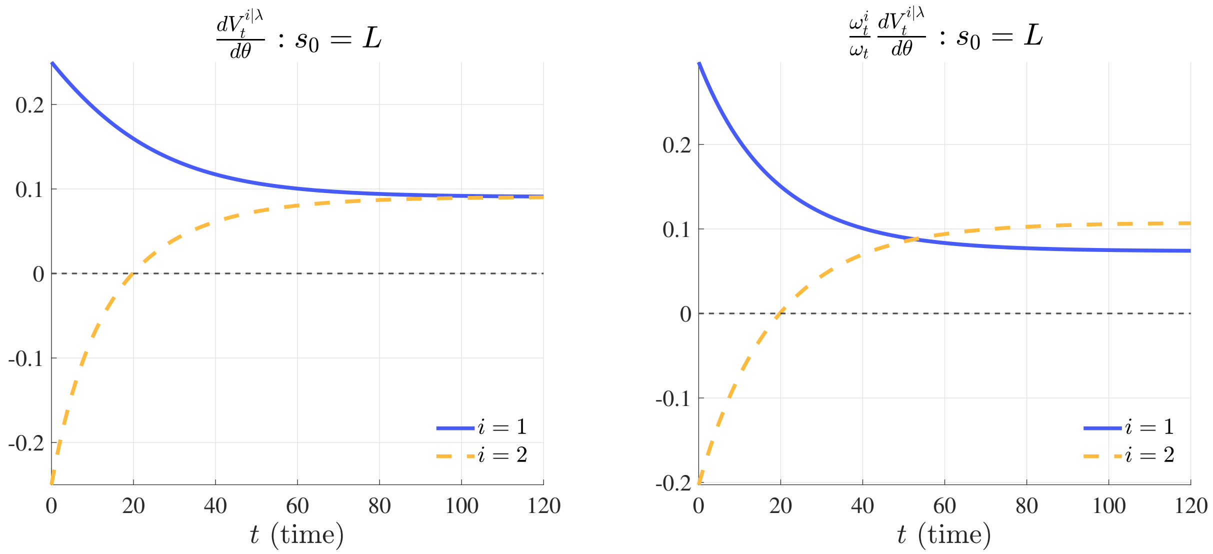

Figure OA-1: Welfare Decomposition and Lifetime Welfare Gains (Application 1)

Note: The left panel of this figure shows the risk-discounted welfare gains for individual i at date t, $\frac{dV_t^{i|\lambda}}{d\theta}$. The right panel of this figure shows the risk-discounted welfare gains for individual i at date t relative to the average, $\frac{\omega_t^i}{\omega_t}\frac{dV_t^{i|\lambda}}{d\theta}$

$\frac{\omega_t^i}{\omega_t}\frac{dV_t^{i|\lambda}}{d\theta}$ does not converge to the same value for both individuals, Ξ_t^{RD} is non-zero (negative) in the long-run. Intuitively, while the long-run welfare gains of the policy at date t are positive and equal for both individuals at date t, such gains are valued more by the individual with a higher endowment at the time of the assessment, since this individual values future consumption in the future relatively more — see the dynamic weights in the left panel of Figure 2. And since the individual with the higher endowment at the time of the assessment also features a lower individual weight ω^i, this logic makes Ξ_t^{RD} negative in the long run.

D.2 Application 2

D.2.1 Deterministic Earnings

The optimal consumption-labor decision for each individual i is given by

$$(1-\tau)\, w^i \frac{\partial u\left(c^i, n^i\right)}{\partial c^i} + \frac{\partial u\left(c^i, n^i\right)}{\partial n^i} = 0. \tag{OA1}$$

Given the assumed preferences $u\left(c,n\right) = \frac{1}{1-\gamma}\left(c - \alpha\frac{n^\sigma}{\sigma}\right)^{1-\gamma}$, equation (OA1) defines a labor supply function

$$n^i\left(\tau\right) = \left(\frac{\left(1-\tau\right) w^i}{\alpha}\right)^{\frac{1}{\sigma-1}},$$

which allows us to express the demogrant as $g(\tau) = \tau \frac{1}{I} \sum_i w^i n^i(\tau)$, which in turn implies that

$$\frac{dg}{d\tau} = \frac{1}{I} \left(\sum_i w^i n^i + \tau \sum_i w^i \frac{dn^i}{d\tau} \right).$$

We can express individual lifetime welfare gains $\frac{dV^i}{d\tau}$ as

$$\begin{aligned}
\frac{dV^i}{d\tau} &= \frac{\partial u(c^i, n^i)}{\partial c^i} \frac{dc^i}{d\tau} + \frac{\partial u(c^i, n^i)}{\partial n^i} \frac{dn^i}{d\tau} \\
&= \frac{\partial u(c^i, n^i)}{\partial c^i} \left(-w^i n^i + (1-\tau) w^i \frac{dn^i}{d\tau} + \frac{dg}{d\tau} \right) + \frac{\partial u(c^i, n^i)}{\partial n^i} \frac{dn^i}{d\tau} \\
&= \frac{\partial u(c^i, n^i)}{\partial c^i} \left(-w^i n^i + \frac{dg}{d\tau} \right),
\end{aligned}$$

which corresponds to equation (16) in the text, using consumption as lifetime welfare numeraire: $\lambda^i = \frac{\partial u(c^i, n^i)}{\partial c^i}$.

Hence, in this economy

$$\frac{dW^\lambda}{d\tau} = \sum_i \omega^i \frac{dV^{i|\lambda}}{d\tau} = \underbrace{\sum_i \frac{dV^{i|\lambda}}{d\theta}}_{\Xi^E} + \underbrace{\mathbb{C}ov_i^\Sigma \left[\omega^i, \frac{dV^{i|\lambda}}{d\theta} \right]}_{\Xi^{RD}},$$

where $\omega^i = \frac{\frac{\partial \mathcal{W}}{\partial V^i} \frac{\partial u(c^i, n^i)}{\partial c^i}}{\frac{1}{I} \sum_i \frac{\partial \mathcal{W}}{\partial V^i} \frac{\partial u(c^i, n^i)}{\partial c^i}}$ (with $\frac{\partial \mathcal{W}}{\partial V^i} = 1$), and where

$$\begin{aligned}
\Xi^E &= \sum_i \left(-w^i n^i + \frac{dg}{d\tau} \right) = \tau \sum_i w^i \frac{dn^i}{d\tau} \\
\Xi^{RD} &= \mathbb{C}ov_i^\Sigma \left[\omega^i, -w^i n^i + \frac{dg}{d\tau} \right] = \mathbb{C}ov_i^\Sigma \left[\omega^i, -w^i n^i \right],
\end{aligned}$$

which corresponds to equation (17) in the text.

D.2.2 Random Earnings

The optimal consumption-labor decision for each individual i is identical to the deterministic case for a given realization of s. Hence,

$$(1-\tau) w^i(s) \frac{\partial u(c^i(s), n^i(s))}{\partial c^i(s)} + \frac{\partial u(c^i(s), n^i(s))}{\partial n^i(s)} = 0. \quad \text{(OA2)}$$

Hence, the labor supply function in state s is given by

$$n^i(s) = \left(\frac{(1-\tau) w^i(s)}{\alpha}\right)^{\frac{1}{\sigma-1}},$$

where the demogrant, which the same regardless of s by virtue of the symmetry assumptions, is $g(\tau) = \tau \frac{1}{I} \sum_i w^i(s) n^i(s)$, which in turn implies that

$$\frac{dg}{d\tau} = \frac{1}{I}\left(\sum_i w^i(s) n^i(s) + \tau \sum_i w^i(s) \frac{dn^i(s)}{d\tau}\right).$$

We can express individual lifetime welfare gains $\frac{dV^i}{d\tau}$ as

$$\begin{aligned}
\frac{dV^i}{d\tau} &= \sum_s \pi(s) \left(\frac{\partial u(c^i(s), n^i(s))}{\partial c^i(s)} \frac{dc^i(s)}{d\tau} + \frac{\partial u(c^i(s), n^i(s))}{\partial n^i} \frac{dn^i(s)}{d\tau}\right) \\
&= \sum_s \pi(s) \frac{\partial u(c^i(s), n^i(s))}{\partial c^i(s)} \left(\frac{dc^i(s)}{d\tau} + \frac{\frac{\partial u\left(c^i(s), n^i(s)\right)}{\partial n^i}}{\frac{\partial u(c^i(s), n^i(s))}{\partial c^i(s)}} \frac{dn^i(s)}{d\tau}\right) \\
&= \sum_s \pi(s) \frac{\partial u(c^i(s), n^i(s))}{\partial c^i(s)} \left(-w^i(s) n^i(s) + (1-\tau) w^i(s) \frac{dn^i(s)}{d\tau} + \frac{dg}{d\tau} + \frac{\frac{\partial u\left(c^i(s), n^i(s)\right)}{\partial n^i}}{\frac{\partial u(c^i(s), n^i(s))}{\partial c^i(s)}} \frac{dn^i(s)}{d\tau}\right) \\
&= \sum_s \pi(s) \frac{\partial u(c^i(s), n^i(s))}{\partial c^i(s)} \left(-w^i(s) n^i(s) + \frac{dg}{d\tau}\right),
\end{aligned}$$

which corresponds to equation (18) in the text, using perpetual consumption as lifetime welfare numeraire: $\lambda^i = \sum_s \pi(s) \frac{\partial u\left(c^i(s), n^i(s)\right)}{\partial c^i(s)}$.

Given the symmetry assumptions, in this economy

$$\frac{dW^\lambda}{d\tau} = \sum_i \omega^i \frac{dV^{i|\lambda}}{d\tau} = \sum_i \frac{dV^{i|\lambda}}{d\theta} = \Xi^E,$$

since $\omega^i = \frac{\frac{\partial \mathcal{W}}{\partial V^i} \sum_s \pi(s) \frac{\partial u\left(c^i(s), n^i(s)\right)}{\partial c^i(s)}}{\frac{1}{I} \sum_i \frac{\partial \mathcal{W}}{\partial V^i} \sum_s \pi(s) \frac{\partial u\left(c^i(s), n^i(s)\right)}{\partial c^i(s)}}$ (with $\frac{\partial \mathcal{W}}{\partial V^i} = 1$) is identical across individuals, and where

$$\frac{dV^{i|\lambda}}{d\tau} = \sum_s \omega^i(s) \frac{dV_1^{i|\lambda}(s)}{d\theta}, \quad \text{where} \quad \frac{dV_1^{i|\lambda}(s)}{d\theta} = -w^i(s) n^i(s) + \frac{dg}{d\tau}.$$

Therefore

$$\frac{dW^\lambda}{d\tau} = \Xi^E = \underbrace{\sum_s \omega_1(s) \sum_i \frac{dV_1^{i|\lambda}(s)}{d\theta}}_{\Xi^{AE}} + \underbrace{\sum_s \omega_1(s) \mathbb{C}ov_i^\Sigma \left[\frac{\omega_1^i(s)}{\omega_1(s)}, \frac{dV_1^{i|\lambda}(s)}{d\theta}\right]}_{\Xi^{RS}},$$

where $\omega_1^i(s) = \frac{\pi(s)\frac{\partial u\left(c^i(s),n^i(s)\right)}{\partial c^i(s)}}{\sum_s \pi(s)\frac{\partial u\left(c^i(s),n^i(s)\right)}{\partial c^i(s)}}$, $\omega_1^i(s) = \frac{1}{I}\sum_i \omega_1^i(s)$, and where

$$\Xi^{AE} = \tau \sum_s \omega_1(s) \sum_i w^i(s) \frac{dn^i(s)}{d\tau}$$
$$\Xi^{RS} = \sum_s \omega_1(s) \mathbb{C}ov_i^{\Sigma}\left[\frac{\omega_1^i(s)}{\omega_1(s)}, -w^i(s) n^i(s)\right],$$

which corresponds to equation (19) in the text.

D.3 Application 3

In addition to market clearing, an equilibrium in this economy is characterized by i) the borrowing/saving optimality conditions for both individuals:

$$u'\left(c_0^i\right) q_0 - \beta \sum_s \pi(s) u'\left(c_1^i(s)\right) = \eta^i,$$

where $\eta^i \geq 0$ denotes the Lagrange multiplier in the borrowing constraint (with $\eta^2 = 0$), and ii) the investment optimality condition for investors:

$$u'\left(c_0^1\right) \Upsilon'\left(k_0^1\right) - \beta \sum_s \pi(s) u'\left(c_1^1(s)\right) z(s) = 0.$$

Provided that the returns to investment are sufficiently attractive (which we always assume), the investor's borrowing constraint binds whenever $\bar{b}$ is sufficiently low, but ceases to bind at a level of $\bar{b}$ we denote by b^u.

We can express individual lifetime welfare gains $\frac{dV^i}{d\bar{b}}$ as

$$\frac{dV^i}{d\bar{b}} = u'\left(c_0^i\right) dc_0^i - \beta \sum_s \pi(s) u'\left(c_1^i(s)\right) dc_1^i(s),$$

where consumption changes are given by

$$dc_0^i = \frac{dq_0}{d\bar{b}} b_1^i + q_0 \frac{db_1^i}{d\bar{b}} - \Upsilon'\left(k_0^i\right) \frac{dk_0^i}{d\bar{b}}$$
$$dc_1^i(s) = z(s) \frac{dk_0^i}{d\bar{b}} - \frac{db_1^i}{d\bar{b}}.$$

Hence, normalized individual lifetime welfare gains take the form

$$\frac{dV^{i|\lambda}}{d\bar{b}} = \frac{\frac{dV^i}{d\bar{b}}}{\lambda^i} = \omega_0^i dc_0^i + \omega_1 \sum_s \omega_1^i(s) dc_1^i(s),$$

where $\omega_0^i = \frac{u'\left(c_0^i\right)}{\lambda^i}$ and $\omega_1^i = \frac{\beta\sum_s \pi(s)u'\left(c_1^i(s)\right)}{\lambda^i}$ with $\lambda^i = u'\left(c_0^i\right) + \beta\sum_s \pi\left(s\right)u'\left(c_1^i\left(s\right)\right)$ and where $\omega_1^i\left(s\right) = \frac{\pi(s)u'\left(c_1^i(s)\right)}{\sum_s \pi(s)u'\left(c_1^i(s)\right)}$.

Hence, in this economy

$$\frac{dW^\lambda}{d\bar{b}} = \sum_i \omega^i \frac{dV^{i|\lambda}}{d\bar{b}} = \sum_i \frac{dV^{i|\lambda}}{d\bar{b}} + \mathbb{C}ov_i^\Sigma\left[\omega^i, \frac{dV^{i|\lambda}}{d\bar{b}}\right],$$

where $\omega^i = \frac{\frac{\partial\mathcal{W}}{\partial V^i}\left(u'\left(c_0^i\right)+\beta\sum_s \pi(s)u'\left(c_1^i(s)\right)\right)}{\frac{1}{I}\sum_i \frac{\partial\mathcal{W}}{\partial V^i}\left(u'\left(c_0^i\right)+\beta\sum_s \pi(s)u'\left(c_1^i(s)\right)\right)}$ with $\frac{\partial\mathcal{W}}{\partial V^i} = 1$. Moreover

$$\Xi^E = \Xi^{AE} + \Xi^{RS} + \Xi^{IS},$$

where

$$\begin{aligned}
\Xi^{AE} &= \omega_0 \sum_i \frac{dc_0^i}{d\bar{b}} + \omega_1 \sum_s \omega_1\left(s\right) \sum_i \frac{dc_1^i\left(s\right)}{d\bar{b}} \\
\Xi^{RS} &= \omega_1 \sum_s \omega_1\left(s\right) \mathbb{C}ov_i^\Sigma\left[\frac{\omega_1^i\left(s\right)}{\omega_1\left(s\right)}, \frac{dc_1^i\left(s\right)}{d\bar{b}}\right] \\
\Xi^{IS} &= \omega_0 \mathbb{C}ov_i^\Sigma\left[\frac{\omega_0^i}{\omega_0}, \frac{dc_0^i}{d\bar{b}}\right] + \omega_1 \mathbb{C}ov_i^\Sigma\left[\frac{\omega_1^i}{\omega_1}, \sum_s \omega_1^i\left(s\right)\frac{dc_1^i\left(s\right)}{d\bar{b}}\right],
\end{aligned}$$

where $\omega_1^i = \frac{1}{I}\sum_i \omega_1^i$ and $\omega_1^i\left(s\right) = \frac{1}{I}\sum_i \omega_1^i\left(s\right)$. Note that the impact of the perturbation on aggregate consumption, which is the key input into the aggregate efficiency component, is given by

$$\begin{aligned}
\sum_i \frac{dc_0^i}{d\bar{b}} &= -\Upsilon'\left(k_0^1\right)\frac{dk_0^1}{d\bar{b}} \\
\sum_i \frac{dc_1^i\left(s\right)}{d\bar{b}} &= z\left(s\right)\frac{dk_0^1}{d\bar{b}}.
\end{aligned}$$

Note that we can use individual optimality (i.e., the envelope theorem) to express lifetime welfare gains as

$$\begin{aligned}
\frac{dV^i}{d\bar{b}} &= u'\left(c_0^i\right)dc_0^i - \beta\sum_s \pi\left(s\right)u'\left(c_1^i\left(s\right)\right)dc_1^i\left(s\right) \\
&= u'\left(c_0^i\right)\left(\frac{dq_0}{d\bar{b}}b_1^i + q_0\frac{db_1^i}{d\bar{b}} - \Upsilon'\left(k_0^i\right)\frac{dk_0^i}{d\bar{b}}\right) - \beta\sum_s \pi\left(s\right)u'\left(c_1^i\left(s\right)\right)\left(z\left(s\right)\frac{dk_0^i}{d\bar{b}} - \frac{db_1^i}{d\bar{b}}\right) \\
&= \underbrace{\left(u'\left(c_0^i\right)q_0 - \beta\sum_s \pi\left(s\right)u'\left(c_1^i\left(s\right)\right)\right)\frac{db_1^i}{d\bar{b}}}_{\text{Direct Effect}} + \underbrace{u'\left(c_0^i\right)\frac{dq_0}{d\bar{b}}b_1^i}_{\text{Distributive Pecuniary Effect}},
\end{aligned}$$

where the direct effect is weakly positive for investors and zero for lenders, and the distributive pecuniary effect is negative for investors and positive for lenders, although zero sum in the aggregate at date 0 since $\sum_i \frac{dq_0}{db} b_1^i = 0$ (Dávila and Korinek, 2018). While this formulation is useful to understand how individual lifetime utility changes, it is not useful to decompose welfare assessments in the way introduced in this paper.

E Generalized Welfare Criteria: DS-planners

Here, we leverage the welfare decomposition to systematically construct non-welfarist welfare criteria based on individual, dynamic, and stochastic weights. This approach allows us to formalize normative objectives that isolate specific components of the welfare decomposition. Because this normative approach entails defining weights for each time and history, for each individual, we say it is based on Dynamic Stochastic Generalized Social Marginal Welfare Weights (dynamic–stochastic weights or DS-weights, for short). These results have the potential to allow for disciplined discussions about the mandates of independent technocratic institutions (central banks, financial regulators, other regulatory agencies, etc.).

DS-planners: Definition. We begin by formally defining desirable perturbations for a planner who adopts DS-weights, a DS-planner.

Definition. *(Desirable perturbation for a DS-planner) A DS-planner finds a perturbation desirable (undesirable) when* $\frac{dW^{DS}}{d\theta} > (<)\, 0$*, where*

$$\frac{dW^{DS}}{d\theta} = \sum_i \omega^i \sum_t \omega_t^i \sum_{s^t} \omega_t^i \left(s^t\right) \frac{dV_t^{i|\lambda}\left(s^t\right)}{d\theta}, \qquad \text{(OA3)}$$

where $\frac{dV_t^{i|\lambda}(s^t)}{d\theta}$ *denotes the history welfare gains at history* s^t *in units of the history welfare numeraire, defined in equation (6), and where* $\omega^i > 0$*,* $\omega_t^i > 0$*, and* $\omega_t^i\left(s^t\right) > 0$ *define individual, dynamic, and stochastic weights that can potentially be functions of outcomes.*

Since $\frac{dV_t^{i|\lambda}(s^t)}{d\theta}$ is expressed in units of the history welfare numeraire and because we require that $\sum_{s^t} \omega_t^i\left(s^t\right) = 1$, the stochastic weight $\omega_t^i\left(s^t\right)$ defines a marginal rate of substitution between a unit of history welfare numeraire at history s^t and a unit of history welfare numeraire across all date t histories for individual i, as in the welfarist case. The dynamic weight ω_t^i defines a marginal rate of substitution between a unit of history welfare numeraire across all date t histories and, implicitly, a unit of lifetime welfare numeraire for individual i.[17] The individual weight ω^i defines how a DS-

[17] Any choice of weights in which $\sum_t \omega_t^i = 1$ and $\sum_{s^t} \omega_t^i\left(s^t\right) = 1$ ensures that interpersonal comparisons are made in a common unit (or lifetime welfare numeraire). It is nonetheless possible to make meaningful comparisons when dynamic weights do not add up to 1 over time.

planner trades off lifetime welfare gains across individuals. The product $\tilde{\omega}_t^i(s^t) = \omega^i \omega_t^i \omega_t^i(s^t)$ defines a dynamic-stochastic weight for individual i.[18]

Unlike the welfarist approach — which takes a social welfare function as primitive — welfare assessments by DS-planners are defined in marginal form. In that sense, DS-planners extend the generalized weight approach in Saez and Stantcheva (2016) to dynamic stochastic environments. Formally, while that paper considers welfare objectives that directly define the individual weight ω^i, DS-planners also define (potentially non-welfarist) dynamic and stochastic weights for each individual. The Online Appendix shows how to equivalently define DS-planners in terms of instantaneous social welfare functions with generalized (endogenous) welfare weights and further relates the results of this section to those in Saez and Stantcheva (2016).

DS-planners can be useful to both provide analytical characterizations and to characterize and compute optimal policies guided by particular components of the welfare decomposition introduced in this paper. Lemma 1 trivially implies that every welfarist planner is a DS-planner, while the converse is not true, as we illustrate next.

AE/AR/NR Pseudo-welfarist DS-planners. Starting from equation (OA3), it is evident that welfare assessments for DS-planners can be decomposed into Ξ^{AE}, Ξ^{RS}, Ξ^{IS}, and Ξ^{RD} components, using the same definitions introduced in (10) and (11). Moreover, Proposition 5a) implies that it is possible to construct welfare objectives in which $\Xi^{RD} = 0$, $\Xi^{IS} = 0$, or $\Xi^{RS} = 0$ by choosing individual, dynamic, or stochastic weights that are invariant across individuals.

First, we focus on pseudo-welfarist DS-planners. These planners justify using particular components of the welfare decomposition of a welfarist planner as the welfare assessment of a particular DS-planner by making individual, dynamic, and/or stochastic weights equal to their cross-sectional welfarist average.

Definition. *(Pseudo-welfarist AE/AR/NR DS-planners) AE (aggregate efficiency), AR (aggregate efficiency/risk-sharing), and NR (no-redistribution) pseudo-welfarist DS-planners are characterized by the normalized weights:*

$$\omega^{i,AE} = 1, \ \omega_t^{i,AE} = \omega_t^{\mathcal{W}}, \ \textit{and} \ \omega_t^{i,AE}\left(s^t\right) = \omega_t^{\mathcal{W}}\left(s^t\right) \qquad \textit{AE Planner} \qquad \text{(OA4)}$$

$$\omega^{i,AR} = 1, \ \omega_t^{i,AR} = \omega_t^{\mathcal{W}}, \ \textit{and} \ \omega_t^{i,AR}\left(s^t\right) = \omega_t^{i,\mathcal{W}}\left(s^t\right) \qquad \textit{AR Planner} \qquad \text{(OA5)}$$

$$\omega^{i,NR} = 1, \ \omega_t^{i,NR} = \omega_t^{i,\mathcal{W}}, \ \textit{and} \ \omega_t^{i,NR}\left(s^t\right) = \omega_t^{i,\mathcal{W}}\left(s^t\right) \qquad \textit{NR Planner} \qquad \text{(OA6)}$$

[18] Earlier versions of this paper defined desirable perturbations directly in terms of DS-weights, as in

$$\frac{dW^{DS}}{d\theta} = \sum_i \sum_t \sum_{s^t} \tilde{\omega}_t^i\left(s^t\right) \frac{dV_t^{i|\lambda}\left(s^t\right)}{d\theta},$$

subsequently multiplicatively decomposing such weights. Both formulations are equivalent.

where $\omega_t^{i,\mathcal{W}}$ *and* $\omega_t^{i,\mathcal{W}}(s^t)$ *are dynamic and stochastic weights for the welfarist planner with social welfare function* $\mathcal{W}(\cdot)$*, and where* $\omega_t^{\mathcal{W}} = \frac{1}{I}\sum_i \omega_t^{i,\mathcal{W}}$ *and* $\omega_t^{\mathcal{W}}(s^t) = \frac{1}{I}\sum_i \omega_t^{i,\mathcal{W}}(s^t)$ *are their cross-sectional averages.*

Pseudo-welfarist planners are constructed so that specific (sums of) components of the welfare decomposition for a given welfarist planner can be interpreted as welfare assessments for particular DS-planners, as we formalize in Proposition 8.[19]

Proposition 8. *(Relation between Welfarist Planners and Pseudo-welfarist AE/AR/NR DS-planners)*

a) *[AE] The aggregate efficiency component* Ξ^{AE} *for a welfarist planner can be interpreted as the welfare assessment of an AE pseudo-welfarist DS-planner, defined in* (OA4)*, for whom* $\Xi^{RS} = \Xi^{IS} = \Xi^{RD} = 0$.

b) *[AR] The sum of aggregate efficiency and risk-sharing components* $\Xi^{AE} + \Xi^{RS}$ *for a welfarist planner can be interpreted as the welfare assessment of an AR pseudo-welfarist DS-planner, defined in* (OA5)*, for whom* $\Xi^{IS} = \Xi^{RD} = 0$.

c) *[NR] The efficiency component* Ξ^{E} *for a welfarist planner can be interpreted as the welfare assessment of a NR pseudo-welfarist DS-planner, defined in* (OA6)*, for whom* $\Xi^{RD} = 0$.

Proof. a) The welfare assessment for the AE pseudo-welfarist DS-planner corresponds to

$$\frac{dW^{AE}}{d\theta} = \sum_t \omega_t^{\mathcal{W}} \sum_{s^t} \omega_t^{\mathcal{W}}\left(s^t\right) \sum_i \frac{dV_t^{i|\lambda}(s^t)}{d\theta},$$

where $\Xi^{RS} = \Xi^{IS} = \Xi^{RD} = 0$, following Proposition 5a).

b) The welfare assessment for the AE pseudo-welfarist DS-planner corresponds to

$$\frac{dW^{AR}}{d\theta} = \sum_t \omega_t^{\mathcal{W}} \sum_{s^t} \omega_t^{\mathcal{W}}\left(s^t\right) \sum_i \frac{dV_t^{i|\lambda}(s^t)}{d\theta} + \sum_t \omega_t^{\mathcal{W}} \sum_{s^t} \omega_t^{\mathcal{W}}\left(s^t\right) \mathbb{C}ov_i^{\Sigma}\left[\frac{\omega_t^{i,\mathcal{W}}(s^t)}{\omega_t^{\mathcal{W}}(s^t)}, \frac{dV_t^{i|\lambda}(s^t)}{d\theta}\right],$$

where $\Xi^{IS} = \Xi^{RD} = 0$, following Proposition 5a).

c) The welfare assessment for the AE pseudo-welfarist DS-planner corresponds to

$$\frac{dW^{NR}}{d\theta} = \sum_t \omega_t^{\mathcal{W}} \sum_{s^t} \omega_t^{\mathcal{W}}\left(s^t\right) \sum_i \frac{dV_t^{i|\lambda}(s^t)}{d\theta} + \sum_t \omega_t^{\mathcal{W}} \sum_{s^t} \omega_t^{\mathcal{W}}\left(s^t\right) \mathbb{C}ov_i^{\Sigma}\left[\frac{\omega_t^{i,\mathcal{W}}(s^t)}{\omega_t^{\mathcal{W}}(s^t)}, \frac{dV_t^{i|\lambda}(s^t)}{d\theta}\right]$$
$$+ \sum_t \omega_t \mathbb{C}ov_i^{\Sigma}\left[\frac{\omega_t^{i,\mathcal{W}}}{\omega_t^{\mathcal{W}}}, \frac{dV_t^{i|\lambda}}{d\theta}\right],$$

[19] The NR pseudo-welfarist planner is equivalent to using a Kaldor-Hicks criterion. This planner is non-welfarist and non-paternalistic — see Remark 5.

where $\Xi^{RD} = 0$, following Proposition 5a). □

We conclude this section with two remarks.

Remark 5. *(Paternalistic vs. Non-paternalistic DS-planners)* DS-planners with non-welfarist dynamic and stochastic weights are paternalistic since their welfare assessments are not based on individual lifetime welfare gains. For instance, those planners may conclude that a perturbation that individuals find Pareto-improving is undesirable. Similarly, the components of the welfare decomposition are based on the weights used by the DS-planner, not those reflecting individual preferences. Therefore, welfare assessments that do not value intertemporal-sharing or risk-sharing as individuals do will be paternalistic.[20] Importantly, the definition of DS-planners in equation (OA3) respects individual intratemporal preferences since it uses $\frac{dV_t^{i|\lambda}(s^t)}{d\theta}$ as an input for the welfare assessment, although this could be relaxed.

Remark 6. *(Impossibility of defining specific pseudo-welfarist DS-planners)* It is not possible to define pseudo-welfarist DS-planners for whom exclusively the risk-sharing and intertemporal-sharing components are zero. Ensuring that $\Xi^{RS} = \Xi^{IS} = 0$ requires using dynamic and stochastic weights that are identical across individuals, which would impact Ξ^{RD}. A similar logic applies to other components of the welfare decomposition. It is nonetheless possible to define DS-planners that are not pseudo-welfarist but that, for instance, exclusively value aggregate efficiency and redistribution, as we show below.

General DS-planners. While above we focus on pseudo-welfarist DS-planners, it is straightforward to define DS-planners that are not pseudo-welfarist. In general, Proposition 5a) provides the recipe to define planners for whom specific components of the welfare decomposition are zero. For instance, one could choose the following weights to define an AE DS-planner:

$$\omega^{i,AE}(s_0) = 1, \quad \omega_t^{i,AE}(s_0) = \beta^t, \quad \text{and} \quad \omega_t^{i,AE}\left(s^t\right) = \pi_t\left(s^t\right),$$

for some β, plausibly $\beta = \frac{1}{I}\sum_i \beta^i$. A similar logic can be used to define general (non-pseudo-welfarist) AR and NR planners. Table OA-1 summarizes which components of the welfare decomposition are zero for general DS-planners. Non-pseudo-welfarist DS-planners may be helpful in particular applications, partly because they may be easier to operationalize.

[20]Welfare assessments by non-welfarist DS-planners (in fact, by non-utilitarian planners) introduce an independent dimension of time inconsistency. There is scope to further explore the time inconsistency of welfare assessments.

Table OA-1: Summary of DS-Planners

DS-Planners	Ξ^{AE}	Ξ^{RS}	Ξ^{IS}	Ξ^{RD}
Aggregate Efficiency (AE)	✓	$=0$	$=0$	$=0$
Aggregate Efficiency/Risk-Sharing (AR)	✓	✓	$=0$	$=0$
No-Redistribution (NR)	✓	✓	✓	$=0$
Welfarist ($\mathcal{W}$)	✓	✓	✓	✓

Note: This table illustrates the non-zero components of the welfare decomposition for particular DS-planners.

DS-weights vs. Normalized Weights. Earlier versions of this paper defined DS-planners in terms of DS-weights, $\tilde{\omega}_t^i(s^t)$, given by

$$\underbrace{\tilde{\omega}_t^i\left(s^t\right)}_{\text{DS-weight}} = \underbrace{\omega^i\left(s_0\right)}_{\text{individual}} \underbrace{\omega_t^i\left(s_0\right)}_{\text{dynamic}} \underbrace{\omega(s^t)}_{\text{stochastic}},$$

as the primitive to define DS-planners. Up to a choice of units for the (aggregate) welfare assessment, there is a one-to-one relation between both approaches.

Note that these formulations respect intratemporal tradeoffs, by taking $\frac{dV_t^{i|\lambda}(s^t)}{d\theta}$, as defined in (6), as a primitive of the welfare assessment. By redefining $\frac{dV_t^{i|\lambda}(s^t)}{d\theta}$ as a weighted sum — based on generalized weights, chosen by a planner — of changes in consumption and factor supply it is possible to define a welfare objective based on generalized weights at the good/factor-history level. This approach can be used to justify a welfare criterion that exclusively loads on changes in aggregate consumption, disregarding the welfare impact of factor supply.

Institutional Design. A central objective of this paper is to provide a framework to systematically formalize new welfare criteria to assess and conduct policy. This has the potential to guide the design of independent technocratic institutions. In practice, such institutions must be given a "mandate", much like defining a set of normalized weights.

Therefore, a society may want to consider designing independent technocratic institutions that have some normative considerations in their mandate but not others, along the lines of the logic developed in this paper. For instance, the current "dual mandate" (stable prices and maximum employment) of the Federal Reserve (as defined by the 1977 Federal Reserve Act) seems to be better described by an aggregate efficiency DS-planner, rather than a welfarist planner, which would care about cross-sectional considerations. Alternatively, an institution like the Federal Emergency Management Agency (FEMA) has as part of its mandate to "support the Nation in a risk-based, comprehensive emergency management system", which unavoidably involves risk-sharing considerations.

Instantaneous Social Welfare Function Formulation. Section E shows that an approach based on generalized marginal DS-weights defined over history welfare gains allows us to systematically define non-welfarist normative objectives. Here, we show that it is possible to interpret $\frac{dW^{DS}}{d\theta}$, defined in equation (OA3), as the welfare assessment of a planner with an (instantaneous) social welfare function that i) takes as arguments individuals' instantaneous utilities, not lifetime utilities, and ii) features generalized (endogenous) welfare weights.

Formally, a linear instantaneous social welfare function, which we denote by $\mathcal{I}(\cdot)$, is a linear function of individuals' instantaneous utilities, given by

$$\mathcal{I}\left(\left\{u_t^i\left(c_t^i\left(s^t\right),n_t^i\left(s^t\right)\right);s^t\right\}_{i,t,s^t}\right)=\sum_i\sum_t\sum_{s^t}\alpha_t^i\left(s^t\right)u_t^i\left(c_t^i\left(s^t\right),n_t^i\left(s^t\right);s^t\right), \tag{OA7}$$

where the instantaneous Pareto weights $\alpha_t^i(s^t)$ define scalars that are individual-, date-, and history-specific. For any set of DS-weights, there exist instantaneous Pareto weights $\{\alpha_t^i(s^t)\}_{i,t,s^t}$ such that $\frac{dW^{DS}}{d\theta}$, defined in equation (OA3), corresponds to the first-order condition of a planner who maximizes a linear instantaneous social welfare function $\mathcal{I}(\cdot)$ with instantaneous Pareto weights $\alpha_t^i(s^t)=\tilde{\omega}_t^i(s^t;\theta)\,/\frac{\partial u^i(s^t;\theta)}{\partial c_t^i}$, since

$$\frac{d\mathcal{I}(\cdot)}{d\theta}=\sum_i\sum_t\sum_{s^t}\alpha_t^i\left(s^t\right)\frac{\partial u_i\left(s^t\right)}{\partial c_t^i}\frac{dV_t^{i|\lambda}\left(s^t\right)}{d\theta}. \tag{OA8}$$

Moreover, at a local optimum, in which $\frac{dW^{DS}}{d\theta}=0$, there exist instantaneous Pareto weights $\{\alpha_t^i(s^t)\}_{i,t,s^t}$ such that the optimal policy satisfies the first-order condition formula of a linear instantaneous social welfare function $\mathcal{I}(\cdot)$, defined in equation (OA7). The instantaneous Pareto weights in that case are evaluated at the optimum, so $\alpha_t^i(s^t)=\tilde{\omega}_t^i(s^t;\theta^\star)\,/\frac{\partial u_i(s^t;\theta^\star)}{\partial c_t^i}$, where $\theta^\star$ denotes the value of θ at the local optimum.

These results are helpful because they show how to reverse-engineer Pareto weights of a linear instantaneous social welfare function from DS-weights, while guaranteeing that any local optimum can be interpreted as the solution to the maximization of a particular linear instantaneous social welfare function. Because the instantaneous Pareto weights $\alpha_t^i(s^t)$ are evaluated at the optimum $\theta^\star$, they are taken as fixed in the maximization of a linear instantaneous social welfare function. In practice, it is impossible to define the instantaneous Pareto weights $\alpha_t^i(s^t)$ without first having solved for the optimum using our approach that starts with DS-weights as primitives. Relatedly, it is typically impossible to translate DS-weights into instantaneous Pareto weights that are invariant to θ and the rest of the environment. As mentioned above, there is scope to explore further the welfare implications of using social welfare functions directly defined over consumption or factor supply at histories.

DS-PLANNERS

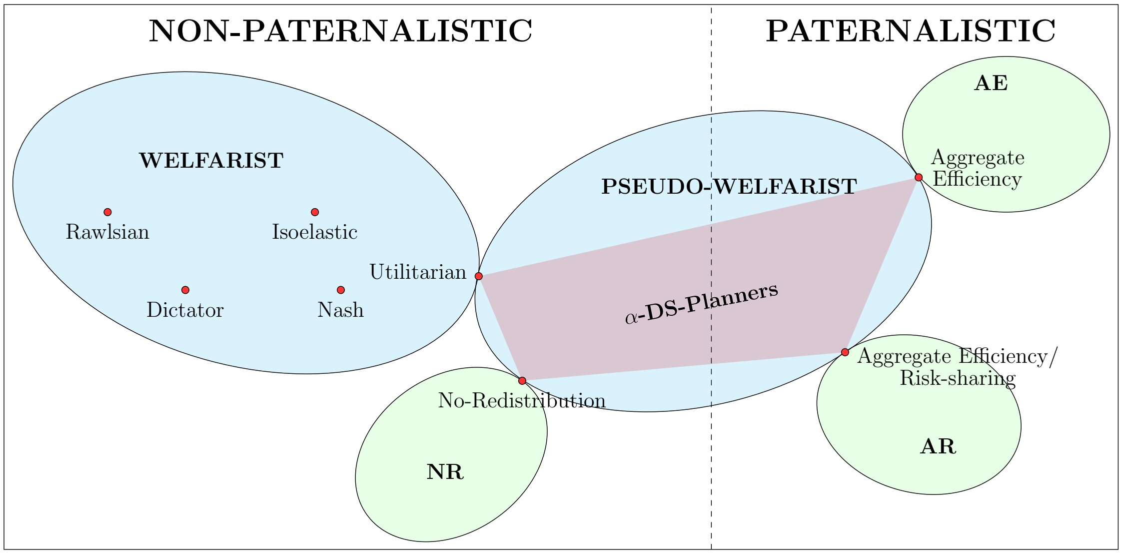

Figure OA-2: DS-planners: Summary

Note: This figure summarizes the relations between the different DS-planners. The vertical dashed line separates non-paternalistic planners from paternalistic planners. All welfarist planners, as well as no-redistribution (NR) planners, are non-paternalistic. Aggregate efficiency (AE) and aggregate efficiency/risk-sharing (AR) planners are paternalistic. Some pseudo-welfarist planners are non-paternalistic (welfarist, NR), while others are paternalistic (AE, AR). In this figure, the $\boldsymbol{\alpha}$-DS-planners are pseudo-welfarist with respect to the utilitarian planner.

$\boldsymbol{\alpha}$-DS-planners. The planners introduced above by no means exhaust the set of new planners that can be defined using DS-weights. In fact, it is possible to define a new planner that spans i) AE, ii) AR, and iii) NR pseudo-welfarist planners, as well as iv) the associated normalized welfarist planner. We refer to this planner as an $\boldsymbol{\alpha}$-DS-planner.

Definition. *($\boldsymbol{\alpha}$-DS-planner: definition) An $\boldsymbol{\alpha}$-DS-planner is a DS-planner for whom the individual, dynamic, and stochastic weights are linear combinations of the weights of a normalized welfarist planner and the weights of an AE pseudo-welfarist planner. An $\boldsymbol{\alpha}$-DS-planner has DS-weights $\omega_t^{i,\mathcal{W},\boldsymbol{\alpha}}(s^t)$ defined by*

$$\begin{aligned}\omega_t^{i,\mathcal{W},\boldsymbol{\alpha}}\left(s^t\right) &= \left(1-\alpha_2\right)\omega_t^{i,\mathcal{W},AE}\left(s^t\right) + \alpha_2\omega_t^{i,\mathcal{W}}\left(s^t\right)\\ \omega_t^{i,\mathcal{W},\boldsymbol{\alpha}} &= \left(1-\alpha_3\right)\omega_t^{i,\mathcal{W},AE} + \alpha_3\omega_t^{i,\mathcal{W}}\\ \omega^{i,\mathcal{W},\boldsymbol{\alpha}} &= \left(1-\alpha_4\right)\omega^{i,\mathcal{W},AE} + \alpha_4\omega^{i,\mathcal{W}},\end{aligned}$$

where $\boldsymbol{\alpha} = (\alpha_2, \alpha_3, \alpha_4)$, and where $\alpha_2 \in [0,1]$, $\alpha_3 \in [0,1]$, $\alpha_4 \in [0,1]$.

Depending on the value of $\boldsymbol{\alpha}$, an $\boldsymbol{\alpha}$-DS-planner behaves as a particular pseudo-welfarist planner or as a combination of pseudo-welfarist planners. When $\boldsymbol{\alpha} = (0,0,0)$, we have a pseudo-welfarist

AE DS-planner; when $\boldsymbol{\alpha} = (1,0,0)$, we have an pseudo-welfarist AR DS-planner; when $\boldsymbol{\alpha} = (1,1,0)$, we have a pseudo-welfarist NR DS-planner; and when $\boldsymbol{\alpha} = (1,1,1)$, we have a welfarist planner. By varying $\boldsymbol{\alpha}$, it is possible to model planners who care about the different components to different degrees. Moreover, estimating $\boldsymbol{\alpha}$ from actual policies in the context of a particular policy problem has the potential to uncover the weights that a particular policymaker attaches in practice to the different components of the welfare decomposition.

F General Welfare Numeraires

In the body of the paper, we immediately adopt a triple of unit-consumption-based welfare numeraires. Here, we proceed to derive the counterpart of Lemma 1 for general welfare numeraires. The main difference with the body of the paper is that we introduce a triple of normalizing factors (lifetime, date, and instantaneous) to allow for general welfare numeraires: λ^i, λ^i_t, and $\lambda^i_t\left(s^t\right)$.

The first step is to express a welfare assessment as

$$\frac{dW}{d\theta} = \sum_i \frac{\partial \mathcal{W}}{\partial V^i} \lambda^i \frac{dV^{i|\lambda}}{d\theta}, \quad \text{where} \quad \frac{dV^{i|\lambda}}{d\theta} = \frac{\frac{dV^i}{d\theta}}{\lambda^i}$$

denotes individual lifetime welfare gains in units of the lifetime welfare numeraire, and where λ^i is the normalizing factor — with units $\frac{\text{individual } i \text{ utils}}{\text{lifetime welfare numeraire}}$ — that allows us to express individual lifetime welfare gains in a common unit. The only restriction when choosing the lifetime welfare numeraire is that λ^i must be strictly positive for all individuals affected by the perturbation.

Next, to meaningfully compare welfare gains at particular dates or histories across individuals in a common unit, we select date and history welfare numeraire for each date and history. Formally, individual lifetime welfare gains in units of the lifetime welfare numeraire, $\frac{dV^{i|\lambda}}{d\theta}$, can be expressed as

$$\frac{dV^{i|\lambda}}{d\theta} = \sum_t \frac{\lambda^i_t}{\lambda^i} \sum_{s^t} \frac{\left(\beta^i\right)^t \pi_t\left(s^t\right) \lambda^i_t\left(s^t\right)}{\lambda^i_t} \frac{dV^{i|\lambda}_t\left(s^t\right)}{d\theta},$$

where

$$\frac{dV^{i|\lambda}_t\left(s^t\right)}{d\theta} = \frac{\frac{\partial u^i_t\left(s^t\right)}{\partial c^i_t}}{\lambda^i_t\left(s^t\right)} \frac{dc^i_t\left(s^t\right)}{d\theta} + \frac{\frac{\partial u^i_t\left(s^t\right)}{\partial n^i_t}}{\lambda^i_t\left(s^t\right)} \frac{dn^i_t\left(s^t\right)}{d\theta} \tag{OA9}$$

denotes normalized history welfare gains at history s^t, expressed in units of the history welfare numeraire, and where $\lambda^i_t\left(s^t\right)$ is the instantaneous normalizing factor — with units $\frac{\text{instantaneous individual } i \text{ utils}}{\text{instantaneous welfare numeraire at } s^t}$ — that allows us to express history welfare gains at history s^t in a common unit at that history and λ^i_t is the date normalizing factor — with units $\frac{\text{individual } i \text{ utils}}{\text{date welfare numeraire at } t}$ — that allows us to express welfare gains at all date-t histories in a common unit at that date. The only restriction when choosing the date and history welfare numeraires is that λ_t^i and $\lambda_t^i(s^t)$ must be strictly positive for all individuals affected by the perturbation at a particular date and history.

In the body of the paper, we assume that λ^i, λ_t^i and $\lambda_t^i(s^t)$ are given by

$$\lambda^i = \sum_t \lambda_t^i \qquad \text{and} \qquad \lambda_t^i = \left(\beta^i\right)^t \sum_{s^t} \pi_t\left(s^t\right)\lambda_t^i\left(s^t\right) \qquad \text{and} \qquad \lambda_t^i\left(s^t\right) = \frac{\partial u_t^i\left(s^t\right)}{\partial c_t^i}, \tag{OA10}$$

which ensures that ω_t^i and $\omega_t^i(s^t)$ define normalized discount factors and risk-neutral probabilities.

In general, the counterparts of the normalized individual, dynamic, stochastic weights in equations (7), (8), and (9) for general welfare numeraires are

$$\omega^i = \frac{\frac{\partial \mathcal{W}}{\partial V^i}\lambda^i}{\frac{1}{I}\sum_i \frac{\partial \mathcal{W}}{\partial V^i}\lambda^i} \qquad \text{and} \qquad \omega_t^i = \frac{\lambda_t^i}{\lambda^i} \qquad \text{and} \qquad \omega_t^i\left(s^t\right) = \frac{\left(\beta^i\right)^t \pi_t\left(s^t\right)\lambda_t^i\left(s^t\right)}{\lambda_t^i},$$

where the interpretation of the weights as marginal rates of substitution is as in the body of the paper, but now using different units.

More generally, the nominal unit (e.g., dollars) or particular commodities or bundles of commodities may also be reasonable choices for welfare numeraires. An alternative choice of lifetime welfare numeraire in models with a single consumption good is date-0 consumption, so $\lambda^i = \frac{\partial u_0^i(s^0)}{\partial c_0^i}$. In this case, the normalized stochastic weights remain unchanged, while the normalized individual and dynamic weights take the form

$$\omega^i = \frac{\frac{\partial \mathcal{W}}{\partial V^i}\frac{\partial u_0^i(s^0)}{\partial c_0^i}}{\frac{1}{I}\sum_i \frac{\partial \mathcal{W}}{\partial V^i}\frac{\partial u_0^i(s^0)}{\partial c_0^i}} \qquad \text{and} \qquad \omega_t^i = \frac{\sum_{s^t}\left(\beta^i\right)^t \pi_t\left(s^t\right)\lambda_t^i\left(s^t\right)}{\frac{\partial u_0^i(s^0)}{\partial c_0^i}}.$$

The main difference with respect to using perpetual consumption as the lifetime welfare numeraire is that now the efficiency component is expressed in terms of willingness-to-pay at date 0. This may be desirable in particular circumstances.

It is worth making three final remarks. First, note that welfare numeraires always exist: at worst, one could choose a bundle of all goods/factors with non-negative marginal utility at a given history, date, or on a lifetime basis. Second, one could potentially pick different numeraires in different dates or histories, but it seems natural to choose a consistent numeraire to yield easily interpretable results. Finally, while the choice of welfare numeraires does not change the directional welfare assessment of a welfarist planner, the interpretation of the elements of the welfare decomposition is contingent on such choice.

G Extensions: Generalized Environments

In this section, we describe how to extend our results to more general environments.

G.1 Heterogeneous Beliefs

Here, we describe how to use the framework introduced in this paper to make welfare assessments in environments with heterogeneous beliefs.[21] To model heterogeneous beliefs, we assume that individual preferences take the form

$$V^i = \sum_t \left(\beta^i\right)^t \sum_{s^t} \pi_t^i\left(s^t\right) u_t^i\left(c_t^i\left(s^t\right), n_t^i\left(s^t\right), s^t\right),$$

where $\pi_t^i\left(s^t\right)$ denotes the beliefs held by individual i over histories, which are now individual-specific. In this case, welfarist welfare assessments (respecting individual beliefs) are as described in the body of the paper, simply using the following normalized weights:

$$\omega^i = \frac{\frac{\partial \mathcal{W}}{\partial V^i} \sum_t \left(\beta^i\right)^t \sum_{s^t} \pi_t^i\left(s^t\right) \frac{\partial u_t^i(s^t)}{\partial c_t^i}}{\frac{1}{I}\sum_i \frac{\partial \mathcal{W}}{\partial V^i} \sum_t \left(\beta^i\right)^t \sum_{s^t} \pi_t^i\left(s^t\right) \frac{\partial u_t^i(s^t)}{\partial c_t^i}}$$

$$\omega_t^i = \frac{\left(\beta^i\right)^t \sum_{s^t} \pi_t^i\left(s^t\right) \frac{\partial u_t^i(s^t)}{\partial c_t^i}}{\sum_t \left(\beta^i\right)^t \sum_{s^t} \pi_t^i\left(s^t\right) \frac{\partial u_t^i(s^t)}{\partial c_t^i}}$$

$$\omega_t^i\left(s^t\right) = \frac{\pi_t^i\left(s^t\right) \frac{\partial u_t^i(s^t)}{\partial c_t^i}}{\sum_{s^t} \pi_t^i\left(s^t\right) \frac{\partial u_t^i(s^t)}{\partial c_t^i}}.$$

Alternatively, a paternalistic planner is a DS-planner — introduced in Section E — who computes welfare using different beliefs than those held by individuals (potentially using a common belief),

[21] A recent literature has explored how to make normative assessments in environments with heterogeneous beliefs. See, among others, Brunnermeier, Simsek and Xiong (2014), Gilboa, Samuelson and Schmeidler (2014), Dávila (2023), Blume et al. (2018), Caballero and Simsek (2019), and Dávila and Walther (2023).

simply computes welfare assessment using the following normalized weights:

$$\omega^i = \frac{\frac{\partial \mathcal{W}}{\partial V^i} \sum_t \left(\beta^i\right)^t \sum_{s^t} \pi_t^{i,P}\left(s^t\right) \frac{\partial u_t^i\left(s^t\right)}{\partial c_t^i}}{\frac{1}{I}\sum_i \frac{\partial \mathcal{W}}{\partial V^i} \sum_t \left(\beta^i\right)^t \sum_{s^t} \pi_t^{i,P}\left(s^t\right) \frac{\partial u_t^i\left(s^t\right)}{\partial c_t^i}}$$

$$\omega_t^i = \frac{\left(\beta^i\right)^t \sum_{s^t} \pi_t^{i,P}\left(s^t\right) \frac{\partial u_t^i\left(s^t\right)}{\partial c_t^i}}{\sum_t \left(\beta^i\right)^t \sum_{s^t} \pi_t^{i,P}\left(s^t\right) \frac{\partial u_t^i\left(s^t\right)}{\partial c_t^i}}$$

$$\omega_t^i\left(s^t\right) = \frac{\pi_t^{i,P}\left(s^t\right) \frac{\partial u_t^i\left(s^t\right)}{\partial c_t^i}}{\sum_{s^t} \pi_t^{i,P}\left(s^t\right) \frac{\partial u_t^i\left(s^t\right)}{\partial c_t^i}},$$

where $\pi_t^{i,P}\left(s^t\right)$ denotes the beliefs used by the planner to compute welfare for individual i at a particular history. In a single belief case, $\pi_t^{i,P}\left(s^t\right) = \pi_t^P\left(s^t\right)$, $\forall i$. See e.g. Dávila and Walther (2023) for an application of this approach to compute optimal leverage regulation.

G.2 General Preferences

G.2.1 Recursive utility: Epstein-Zin Preferences

Here, we describe how to allow for recursive preferences. In particular, we consider the widely used Epstein-Zin preferences, defined recursively as follows:

$$V^i(s) = \left(\left(1-\beta^i\right)\left(u^i\left(c^i(s), n^i(s)\right)\right)^{1-\frac{1}{\psi^i}} + \beta^i\left(\sum_{s'} \pi\left(s'|s\right)\left(V^i\left(s'\right)\right)^{1-\gamma^i}\right)^{\frac{1-\frac{1}{\psi^i}}{1-\gamma^i}}\right)^{\frac{1}{1-\frac{1}{\psi^i}}},$$

where γ^i modulates risk aversion and ψ^i modulates intertemporal substitution. We use s and s' to denote any two recursive states (Ljungqvist and Sargent, 2018).

In this case, we can recursively express the lifetime welfare gains of a perturbation in utils, as follows:

$$\frac{dV^i(s)}{d\theta} = \frac{\partial V^i(s)}{\partial c^i(s)} \frac{du^{i|\lambda}(s)}{d\theta} + \sum_{s'} \frac{\partial V^i(s)}{\partial V^i(s')} \frac{dV^i(s')}{d\theta}, \tag{OA11}$$

where

$$\frac{\partial V^i(s)}{\partial c^i(s)} = \left(1-\beta^i\right)\left(V^i(s)\right)^{\frac{1}{\psi^i}}\left(u^i(s)\right)^{-\frac{1}{\psi^i}} \frac{\partial u^i(s)}{\partial c^i}$$

$$\frac{\partial V^i(s)}{\partial V^i(s')} = \beta^i\left(V^i(s)\right)^{\frac{1}{\psi^i}}\left(\sum_{s'} \pi\left(s'|s\right)\left(V^i\left(s'\right)\right)^{1-\gamma^i}\right)^{\frac{\gamma^i-\frac{1}{\psi^i}}{1-\gamma^i}} \pi\left(s'|s\right)\left(V^i\left(s'\right)\right)^{-\gamma^i},$$

and where

$$\frac{du^{i|\lambda}\left(s\right)}{d\theta}=\frac{dc_{t}^{i}\left(s\right)}{d\theta}+\frac{\frac{\partial V^{i}(s)}{\partial n^{i}(s)}}{\frac{\partial V^{i}(s)}{\partial c^{i}(s)}}\frac{dn_{t}^{i}\left(s\right)}{d\theta}.$$

The structure of equation (OA11) immediately implies that $\frac{dV^{i}(s)}{d\theta}$ can be expressed as a linear transformation of history welfare gains, which in turn guarantees that $\frac{dV^{i}(s)}{d\theta}$ can be written as in Lemma (1). It is easiest to leverage equation (13) to compute normalized weights via state-prices for any date and state, as follows:

$$q^{i}\left(s'|s\right)=\frac{\frac{\partial V_{i}(s)}{\partial c^{i}(s')}}{\frac{\partial V^{i}(s)}{\partial c^{i}(s)}}=\frac{\frac{\partial V^{i}(s)}{\partial V^{i}(s')}\frac{\partial V^{i}(s')}{\partial c^{i}(s')}}{\frac{\partial V^{i}(s)}{\partial c^{i}(s)}}=\beta^{i}\pi\left(s'|s\right)\left(\frac{V^{i}\left(s'\right)}{H\left(s\right)}\right)^{\frac{1}{\psi^{i}}-\gamma^{i}}\left(\frac{c^{i}\left(s'\right)}{c^{i}\left(s\right)}\right)^{-\frac{1}{\psi^{i}}}\frac{\frac{\partial u^{i}(s')}{\partial c^{i}}}{\frac{\partial u^{i}(s)}{\partial c^{i}}},$$

where $H\left(s\right)=\left(\sum_{s'}\pi\left(s'|s\right)\left(V^{i}\left(s'\right)\right)^{1-\gamma^{i}}\right)^{\frac{1}{1-\gamma^{i}}}$. It is straightforward to define DS-weights for even more general preferences, including preferences that are not time-separable or recursive, as we do next.

G.2.2 General Non-separable Preferences

It is possible to consider general non-expected utility non-time separable preferences of the form (we abstract from factor supply only for simplicity, the results extend straightforwardly to that case):

$$V^{i}=U^{i}\left(\left\{c_{t}^{i}\left(s^{t}\right)\right\}_{t,s^{t}}\right).$$

Individual lifetime welfare gains take the form

$$\frac{dV^{i}}{d\theta}=\sum_{t}\sum_{s^{t}}\frac{\partial U^{i}}{\partial c_{t}^{i}\left(s^{t}\right)}\frac{dc_{t}^{i}\left(s^{t}\right)}{d\theta}.$$

From here it is evident that Lemma 1 applies, with normalized weights of the form

$$\omega^{i}=\frac{\frac{\partial\mathcal{W}}{\partial V^{i}}\sum_{t}\sum_{s^{t}}\frac{\partial U^{i}}{\partial c_{t}^{i}(s^{t})}}{\frac{1}{I}\sum_{i}\frac{\partial\mathcal{W}}{\partial V^{i}}\sum_{t}\sum_{s^{t}}\frac{\partial U^{i}}{\partial c_{t}^{i}(s^{t})}}$$

$$\omega_{t}^{i}=\frac{\sum_{s^{t}}\frac{\partial U^{i}}{\partial c_{t}^{i}(s^{t})}}{\sum_{t}\sum_{s^{t}}\frac{\partial U^{i}}{\partial c_{t}^{i}(s^{t})}}$$

$$\omega_{t}^{i}\left(s^{t}\right)=\frac{\frac{\partial U^{i}}{\partial c_{t}^{i}(s^{t})}}{\sum_{s^{t}}\frac{\partial U^{i}}{\partial c_{t}^{i}(s^{t})}}.$$

G.3 Multiple Goods/Factors

Here, we extend the result to economies with multiple goods/factors. We extend the baseline environment by assuming individuals consume $J \geq 1$ goods, indexed by $j \in \mathcal{J} = \{1, \ldots, J\}$, and supply $F \geq 0$ factors, indexed by $f \in \mathcal{F} = \{1, \ldots, F\}$, At all dates and histories. In this case, preferences are given by

$$V^i = \sum_t \left(\beta^i\right)^t \sum_{s^t} \pi_t \left(s^t\right) u_t^i \left(\left\{c_t^{ij}\left(s^t\right)\right\}_j, \left\{n_t^{if}\left(s^t\right)\right\}_f; s^t\right), \qquad \text{(Preferences)}$$

where $c_t^{ij}\left(s^t\right)$ and $n_t^{if}\left(s^t\right)$ respectively denote the consumption of good j and the amount of factor f supplied by individual i at history s^t.

In this case, individual lifetime welfare gains are given by

$$\frac{dV^i}{d\theta} = \sum_t \left(\beta^i\right)^t \sum_{s^t} \pi_t \left(s^t\right) \lambda_t^i \left(s^t\right) dV_t^{i|\lambda}\left(s^t\right),$$

where

$$dV_t^{i|\lambda}\left(s^t\right) = \sum_j \frac{\frac{\partial u_t^i\left(s^t\right)}{\partial c_t^{ij}}}{\lambda_t^i\left(s^t\right)} \frac{dc_t^{ij}\left(s^t\right)}{d\theta} + \sum_f \frac{\frac{\partial u_t^i\left(s^t\right)}{\partial n_t^{if}}}{\lambda_t^i\left(s^t\right)} \frac{dn_t^{if}\left(s^t\right)}{d\theta},$$

which generalizes equation (6) in the text. Given this, Proposition 1 and all the other results follow straightforwardly.

G.4 Perturbations to Probabilities

It is possible to consider perturbations that affect probabilities. Starting from equation (1), we can express $\frac{dV^i}{d\theta}$ as

$$\frac{dV^i}{d\theta} = \sum_t \left(\beta^i\right)^t \sum_{s^t} \left(\pi_t\left(s^t\right)\left(\frac{\partial u_t^i\left(s^t\right)}{\partial c_t^i}\frac{dc_t^i\left(s^t\right)}{d\theta} + \frac{\partial u_t^i\left(s^t\right)}{\partial n_t^i}\frac{dn_t^i\left(s^t\right)}{d\theta}\right) + \frac{d\pi_t\left(s^t\right)}{d\theta} u_i\left(c_t^i\left(s^t\right), n_t^i\left(s^t\right)\right)\right).$$

Hence, the definition of Ξ^{RD} and Ξ^{IS} apply unchanged, with the addition of the new term that includes how the change in probabilities impacts lifetime and date t welfare gains, respectively. The split between aggregate efficiency and risk-sharing now includes an additional term that takes the form

$$\sum_t \omega_t \sum_i \sum_{s^t} \zeta_t^i(s) \frac{d\pi_t\left(s^t\right)}{d\theta}, \quad \text{where} \quad \zeta_t^i = \frac{u_i\left(c_t^i\left(s^t\right), n_t^i\left(s^t\right)\right)}{\sum_{s^t} \pi_t\left(s^t\right) \frac{\partial u_t^i\left(s^t\right)}{\partial c_t^i}}.$$

G.5 Zeros

G.5.1 Zero Weights

In the body of the paper, we exclusively consider social welfare functions in which $\frac{\partial \mathcal{W}}{\partial V^i} > 0$, and implicitly, since we assume that individual marginal utilities of consumption are strictly positive at all times, normalized dynamic and stochastic weights such that $\omega_t^i > 0$ and $\omega_t^i(s^t) > 0$. Here, we present a generalized decomposition that accommodates normalized weights to be zero for particular individuals, dates, or histories. To allow for this possibility, we must appropriately define the set of individuals over which sums and covariances are computed.[22]

Formally, a normalized welfare assessment takes the form:

$$\frac{dW^\lambda}{d\theta} = \sum_i \omega^i \frac{dV^{i|\lambda}}{d\theta} = \underbrace{\sum_{i|\omega^i>0} \frac{dV^{i|\lambda}}{d\theta}}_{\Xi^E} + \underbrace{\mathbb{C}ov^{\Sigma}_{i|\omega^i>0}\left[\omega^i, \frac{dV^{i|\lambda}}{d\theta},\right]}_{\Xi^{RD}},$$

where $\mathbb{C}ov^{\Sigma}_{i|\omega^i>0}\left[\omega^i, \frac{dV^{i|\lambda}}{d\theta}\right] = \sum_i \mathbb{I}\left[\omega^i > 0\right] \mathbb{C}ov_{i|\omega^i>0}\left[\omega^i, \frac{dV^{i|\lambda}}{d\theta}\right]$, where $\mathbb{I}[\cdot]$ denotes an indicator, and where $\omega^i = \frac{\frac{\partial \mathcal{W}}{\partial V^i}\lambda^i}{\frac{1}{\sum_i \mathbb{I}[\omega^i>0]}\sum_{i|\omega^i>0}\frac{\partial \mathcal{W}}{\partial V^i}\lambda^i}$. In this case, both efficiency and redistribution exclusively account for the lifetime welfare gains of those individuals for whom $\omega^i > 0$. Following the same steps as in the baseline case, the efficiency component can be expressed as

$$\Xi^E = \sum_{i|\omega^i>0} \frac{dV^{i|\lambda}}{d\theta} = \sum_t \sum_{i|\omega^i>0} \omega_t^i \frac{dV_t^{i|\lambda}}{d\theta} = \sum_t \omega_t \sum_{i|\omega^i,\omega_t^i>0} \frac{dV_t^{i|\lambda}}{d\theta} + \underbrace{\sum_t \omega_t \mathbb{C}ov^{\Sigma}_{i|\omega^i,\omega_t^i>0}\left[\frac{\omega_t^i}{\omega_t}, \frac{dV_t^{i|\lambda}}{d\theta}\right]}_{\Xi^{IS}},$$

where $\Xi^{AE} = \sum_t \omega_t \Xi_t^{AE}$ and $\Xi^{RS} = \sum_t \omega_t \Xi_t^{RS}$ with

$$\begin{aligned}\sum_{i|\omega^i,\omega_t^i>0} \frac{dV_t^{i|\lambda}}{d\theta} &= \sum_{s^t} \sum_{i|\omega^i,\omega_t^i>0} \omega_t^i\left(s^t\right) \frac{dV_t^{i|\lambda}\left(s^t\right)}{d\theta} \\ &= \underbrace{\sum_{s^t} \omega_t\left(s^t\right) \sum_{i|\omega^i,\omega_t^i,\omega_t^i(s^t)>0} \frac{dV_t^{i|\lambda}\left(s^t\right)}{d\theta}}_{\Xi_t^{AE}} + \underbrace{\sum_{s^t} \omega_t\left(s^t\right) \mathbb{C}ov^{\Sigma}_{i|\omega^i,\omega_t^i,\omega_t^i(s^t)>0}\left[\frac{\omega_t^i\left(s^t\right)}{\omega_t\left(s^t\right)}, \frac{dV_t^{i|\lambda}\left(s^t\right)}{d\theta}\right]}_{\Xi_t^{RS}},\end{aligned}$$

[22] We repeatedly use the fact that

$$\sum_i x^i y^i = \frac{1}{I^+} \sum_{i|x^i>0} x^i \sum_{i|x^i>0} y^i + \sum_{i|x^i>0} \left(x_i - \frac{1}{I^+}\sum_{i|x^i>0} x^i\right)\left(y_i - \frac{1}{I^+}\sum_{i|x^i>0} y^i\right),$$

where $I^+ = \sum_i \mathbb{I}\left[x^i > 0\right]$.

where $\omega_t = \sum_{i|\omega^i,\omega_t^i>0} \omega_t^i$ and $\omega_t(s^t) = \sum_{i|\omega^i,\omega_t^i,\omega_t^i(s^t)>0} \omega_t^i(s^t)$, and where

$$\mathbb{C}ov^{\Sigma}_{i|\omega^i,\omega_t^i>0}\left[\frac{\omega_t^i}{\omega_t}, \frac{dV_t^{i|\lambda}}{d\theta}\right] = \sum_i \mathbb{I}\left[\omega^i, \omega_t^i > 0\right] \mathbb{C}ov_{i|\omega^i,\omega_t^i>0}\left[\frac{\omega_t^i}{\omega_t}, \frac{dV_t^{i|\lambda}}{d\theta}\right].$$

$$\mathbb{C}ov^{\Sigma}_{i|\omega^i,\omega_t^i,\omega_t^i(s^t)>0}\left[\frac{\omega_t^i(s^t)}{\omega_t(s^t)}, \frac{dV_t^{i|\lambda}(s^t)}{d\theta}\right] = \sum_i \mathbb{I}\left[\omega^i, \omega_t^i, \omega_t^i(s^t) > 0\right] \mathbb{C}ov_{i|\omega^i,\omega_t^i,\omega_t^i(s^t)>0}\left[\frac{\omega_t^i(s^t)}{\omega_t(s^t)}, \frac{dV_t^{i|\lambda}(s^t)}{d\theta}\right].$$

In this case, only individuals with positive dynamic or stochastic weights enter into Ξ^{AE}, Ξ^{RS}, and Ξ^{IS}.

It is worth highlighting that Proposition 2a) no longer holds when $\omega^i = 0$ for individuals. For instance, a dictator who exclusively cares about individual $i = 1$, attributes all welfare gains to the efficiency component (actually Ξ^{AE}) for individual 1, but the efficiency component for such a dictator is different from the efficiency component of a utilitarian planner or any other welfarist planner who puts strictly positive weight on all individuals. Propositions 2b) and c), as well as Propositions 4, 5, and Proposition 6a) through d) still hold, but Proposition 6e) also fails.

The central takeaway from these results is that the welfare decomposition must be interpreted only for the individuals for positive weights when i) planners completely disregard the welfare gains by specific individuals, or ii) individuals do not value at all welfare gains at particular dates or states. For instance, the redistribution component for a planner who exclusively cares about individuals $i = 1$ and $i = 2$ is exclusively based on the lifetime welfare gains of these two individuals, disregarding the rest. The same logic applies to the remaining terms of the decomposition.

G.5.2 Zero Welfare Gains

One may be tempted to also condition the covariance decomposition on the welfare gains terms to be non-zero, e.g. $\frac{dV^i}{d\theta} \neq 0$, but this would lead to erroneous conclusions. For instance, it may be that $\frac{dV^{i|\lambda}}{d\theta} = 0$ when $\frac{dV_t^{i|\lambda}}{d\theta} \neq 0$ or $\frac{dV_t^{i|\lambda}(s^t)}{d\theta} \neq 0$, which would yield incorrect results. An implication of always considering all individuals with $\omega^i > 0$, even when $\frac{dV^{i|\lambda}}{d\theta} = 0$, is that Ξ^{RD}, as well as the split of the efficiency among its three constituents will depend on the normalized weights of all individuals in the economy, including those unaffected directly by the perturbation. However, the efficiency component as a whole will not.

G.6 Other considerations

Idiosyncratic/Aggregate States In recursive economies with idiosyncratic (and potentially aggregate) states (i.e., Aiyagari or Krusell-Smith style economies) individuals can be heterogeneous at the time of making a welfare assessment for two different reasons. First, individuals can be heterogeneous ex-ante (e.g., individuals can have different time-invariant preferences or face shocks that come from different distributions). Second, individuals can be heterogeneous ex-post (e.g., individuals can have different endowments or asset holdings at the time of the welfare assessment, even though they face identical problems starting from a given idiosyncratic state). This distinction is important to interpret correctly some of the results in this paper. For instance, 6d) only applies when all individuals are identical because of predetermined reasons and when they all have the same initial state. Formally, ex-ante heterogeneity of either form is captured by the index i in this paper. It is possible to further refine the composition in environments that differentiate between idiosyncratic and aggregate states along the lines of Section H.

Continuum of Individuals, Continuous Time, Continuum of States In order to highlight the differences between averages and signs, we have considered an environment with countable individuals, dates, and histories. It is straightforward to extend the results to environments with a continuum of individuals, continuous time, and a continuum of histories. In fact, earlier versions of this paper included examples of all three cases.

Non-differentiabilities It is possible to generalize the results to environments in which lifetime or instantaneous utilities are not differentiable. For lifetime utilities, it is necessary to consider global assessments, as described in Section I.3. For instantaneous utilities, it is typically possible to incorporate non-differentiabilities using Leibniz rule — see Dávila and Goldstein (2023) for an application.

H Extensions: Subdecompositions and Alternative Decompositions

In this section, we describe how to further decompose the components of the welfare decomposition introduced in this paper. At times, we refer to two properties of covariances:

$$\mathbb{C}ov_i^{\Sigma}\left[x^i, y^i z^i\right] = \mathbb{E}_i\left[y^i\right]\mathbb{C}ov_i^{\Sigma}\left[x^i, z^i\right] + \mathbb{E}_i\left[z^i\right]\mathbb{C}ov_i^{\Sigma}\left[x^i, y^i\right] + \sum_i\left[\left(x^i - \mathbb{E}_i\left[x^i\right]\right)\left(y^i - \mathbb{E}_i\left[y^i\right]\right)\left(z^i - \mathbb{E}_i\left[z^i\right]\right)\right] \tag{OA12}$$

$$\mathbb{C}ov_i^{\Sigma}\left[x^i, y^i\right] = \sum_i\left[\mathbb{C}ov_i\left[x^i, y^i \mid z^i\right]\right] + \mathbb{C}ov_i^{\Sigma}\left[\mathbb{E}_i\left[x^i \mid z^i\right], \mathbb{E}_i\left[y^i \mid z^i\right]\right], \tag{OA13}$$

where X, Y, and Z denote random variables. The first property is established in Bohrnstedt and Goldberger (1969). The second is the Law of Total Covariance, and is standard. Figure OA-3 illustrates the decompositions introduced in Subsections H.4 and H.5.

It is worth highlighting that Lemma 1 implies that any decomposition of welfare assessments

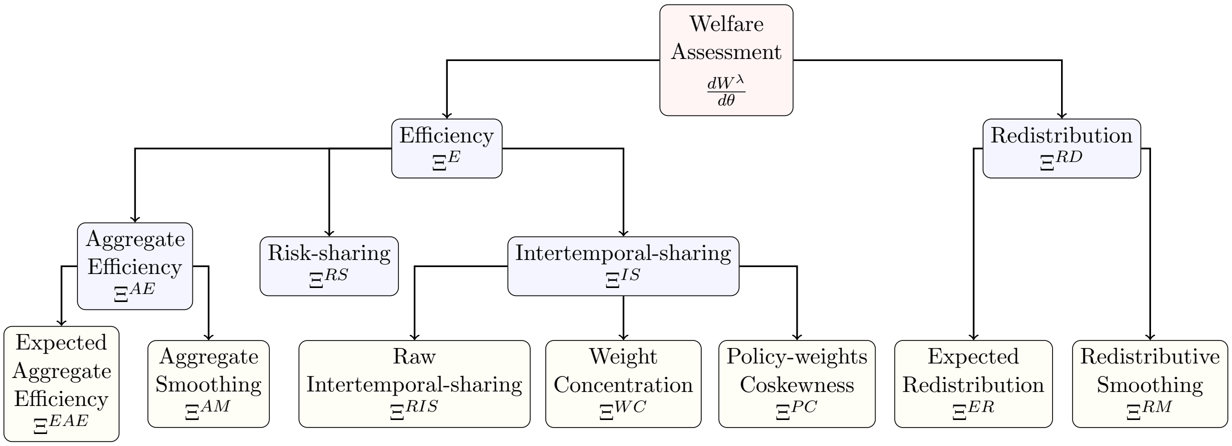

Figure OA-3: Subdecomposition

Note: This figure illustrates how the welfare decomposition can be subdecomposed. Section H describes multiple ways of subdecomposing the welfare decomposition and discusses alternative decompositions.

boils down to defining particular groupings of the triple sum:

$$\frac{dW^{\lambda}}{d\theta}=\sum_{i}\omega^{i}\sum_{t}\omega_{t}^{i}\sum_{s^{t}}\omega_{t}^{i}\left(s^{t}\right)\frac{dV_{t}^{i|\lambda}\left(s^{t}\right)}{d\theta}.$$

H.1 Term Structure and Related Results

Here, we show that the welfare decomposition and each of its components has a term structure. That is, it is possible to attribute welfare gains in the aggregate or for each of the components to particular dates in the future. Formally, note that

$$\frac{dW^{\lambda}}{d\theta}=\sum_{t}\omega_{t}\frac{dW_{t}^{\lambda}}{d\theta}\quad\text{where}\quad\frac{dW_{t}^{\lambda}}{d\theta}=\Xi_{t}^{AE}+\Xi_{t}^{RS}+\Xi_{t}^{IS}+\Xi_{t}^{RD},\tag{OA14}$$

where

$$\Xi_{t}^{AE}=\sum_{s^{t}}\omega_{t}\left(s^{t}\right)\Xi_{t}^{AE}\left(s^{t}\right)\quad\text{where}\quad\Xi_{t}^{AE}\left(s^{t}\right)=\sum_{i}\frac{dV_{t}^{i|\lambda}\left(s^{t}\right)}{d\theta}$$

$$\Xi_{t}^{RS}=\sum_{s^{t}}\omega_{t}\left(s^{t}\right)\Xi_{t}^{RS}\left(s^{t}\right)\quad\text{where}\quad\Xi_{t}^{RS}\left(s^{t}\right)=\mathbb{C}ov_{i}^{\Sigma}\left[\frac{\omega_{t}^{i}\left(s^{t}\right)}{\omega_{t}\left(s^{t}\right)},\frac{dV_{t}^{i|\lambda}\left(s^{t}\right)}{d\theta}\right]$$

$$\Xi_{t}^{IS}=\sum_{s^{t}}\omega_{t}\left(s^{t}\right)\Xi_{t}^{IS}\left(s^{t}\right)\quad\text{where}\quad\Xi_{t}^{IS}\left(s^{t}\right)=\mathbb{C}ov_{i}^{\Sigma}\left[\frac{\omega_{t}^{i}}{\omega_{t}},\frac{\omega_{t}^{i}\left(s^{t}\right)}{\omega_{t}\left(s^{t}\right)}\frac{dV_{t}^{i|\lambda}\left(s^{t}\right)}{d\theta}\right]$$

$$\Xi_{t}^{RD}=\sum_{s^{t}}\omega_{t}\left(s^{t}\right)\Xi_{t}^{RD}\left(s^{t}\right)\quad\text{where}\quad\Xi_{t}^{RD}\left(s^{t}\right)=\mathbb{C}ov_{i}^{\Sigma}\left[\omega^{i},\frac{\omega_{t}^{i}}{\omega_{t}}\frac{\omega_{t}^{i}\left(s^{t}\right)}{\omega_{t}\left(s^{t}\right)}\frac{dV_{t}^{i|\lambda}\left(s^{t}\right)}{d\theta}\right].$$

This formulation shows that a welfare assessment can be interpreted as the discounted sum, using an aggregate discount factor — of date-specific welfare assessments, where each of these date-specific assessments can be further decomposed into aggregate efficiency, risk-sharing, intertemporal-sharing, and redistribution.

Transition vs. Steady State Welfare Gains. Equation (OA14) also allows us to decompose the transition and steady-state impact of perturbations for aggregate assessments and each of the components of the welfare decomposition. Formally, under the assumption that an economy reaches a new steady-state at date $T^{\star}$, it is possible to decompose welfare assessments into transition welfare effects and steady-state welfare effects:

$$\frac{dW^{\lambda}}{d\theta} = \underbrace{\sum_{t=0}^{T^{\star}} \omega_t \frac{dW_t^{\lambda}}{d\theta}}_{\text{transition welfare gains}} + \underbrace{\sum_{t=T^{\star}}^{T} \omega_t \frac{dW_t^{\lambda}}{d\theta}}_{\text{steady-state welfare gains}} .$$

It is worth highlighting that convergence to a new steady-state in terms of allocations does not guarantee convergence of normalized weights. To facilitate comparisons, it seems more natural to report the value of steady-state welfare effects expressed in perpetual units starting at $T^{\star}$, rather than starting at date-0, that is: $\frac{\sum_{t=T^{\star}}^{T} \omega_t \frac{dW_t^{\lambda}}{d\theta}}{\sum_{t=T^{\star}}^{T} \omega_t}$.

Stochastic Structure. Finally note that it is possible to express a welfare assessment as

$$dW^{\lambda} = \sum_t \omega_t \sum_{s^t} \omega_t \left(s^t\right) \left(\Xi_t^{AE}\left(s^t\right) + \Xi_t^{RS}\left(s^t\right) + \Xi_t^{IS}\left(s^t\right) + \Xi_t^{RD}\left(s^t\right)\right). \tag{OA15}$$

This formulation shows that a welfare assessment can be interpreted as the discounted sum, using aggregate time and stochastic discount factors — of history-specific welfare assessments, where each of these history-specific assessments can be further decomposed into aggregate efficiency, risk-sharing, intertemporal-sharing, and redistribution. This formulation allows us to attribute welfare gains due to each of the components of the welfare decomposition to specific histories.

H.2 Individual Structure

Since each of the components of the welfare decomposition can be expressed as a triple-sum (over individuals, dates, and histories), it is also possible to compute the individual contribution of particular individuals to each of the components of the welfare decomposition. Formally, we can write

$$\Xi^{AE} = \sum_i \Xi^{i,AE} \quad \text{where} \quad \Xi^{i,AE} = \sum_t \omega_t \sum_{s^t} \omega_t\left(s^t\right) \frac{dV_t^{i|\lambda}\left(s^t\right)}{d\theta}$$

$$\Xi^{RS} = \sum_i \Xi^{i,RS} \quad \text{where} \quad \Xi^{i,RS} = \sum_t \omega_t \sum_{s^t} \omega_t\left(s^t\right) \left(\frac{\omega_t^i\left(s^t\right)}{\omega_t\left(s^t\right)} - 1\right) \frac{dV_t^{i|\lambda}\left(s^t\right)}{d\theta}$$

$$\Xi^{IS} = \sum_i \Xi^{i,IS} \quad \text{where} \quad \Xi^{i,IS} = \sum_t \omega_t \left(\frac{\omega_t^i}{\omega_t} - 1\right) \frac{dV_t^{i|\lambda}}{d\theta}$$

$$\Xi^{RD} = \sum_i \Xi^{i,RD} \quad \text{where} \quad \Xi^{i,RD} = \left(\omega^i - 1\right) \frac{dV^{i|\lambda}}{d\theta},$$

where, by construction, $\sum_i \xi^{i,AE} = 1$ with $\xi^{i,AE} = \frac{\Xi^{i,AE}}{\Xi^{AE}}$; and analogously for the other three components.

H.3 Reallocation and Growth

In order to separate welfare gains due to reallocation from those due to changes in aggregates, it may be useful to decompose how the changes in the level of consumption (analogously, factor supply) are due to changes in the share of consumption across individuals or to changes in aggregate consumption. Formally, we can define consumption and factor supply shares at a given history by $\chi_{t,c}^i\left(s^t\right) = \frac{c_t^i\left(s^t\right)}{c_t\left(s^t\right)}$, and $\chi_{t,n}^i\left(s^t\right) = \frac{n_t^i\left(s^t\right)}{n_t\left(s^t\right)}$where $c_t\left(s^t\right) = \sum_i c_t^i\left(s^t\right)$ and $n_t\left(s^t\right) = \sum_i n_t^i\left(s^t\right)$. Hence, by applying the product rule, we can express $\frac{dc_t^i\left(s^t\right)}{d\theta}$ and $\frac{dn_t^i\left(s^t\right)}{d\theta}$ as

$$\frac{dc_t^i\left(s^t\right)}{d\theta} = \underbrace{\frac{d\chi_{t,c}^i\left(s^t\right)}{d\theta} c_t\left(s^t\right)}_{=\text{Reallocation}} + \underbrace{\chi_{t,c}^i\left(s^t\right) \frac{dc_t\left(s^t\right)}{d\theta}}_{=\text{Growth}}$$

$$\frac{dn_t^i\left(s^t\right)}{d\theta} = \underbrace{\frac{d\chi_{t,n}^i\left(s^t\right)}{d\theta} n_t\left(s^t\right)}_{=\text{Reallocation}} + \underbrace{\chi_{t,n}^i\left(s^t\right) \frac{dn_t\left(s^t\right)}{d\theta}}_{=\text{Growth}}.$$

Hence, combining these definitions with the definition of history welfare gains in (6) or (OA9), it is possible to subdecompose each of the components of the welfare decomposition into terms that capture reallocation or aggregate growth of consumption or factor supply.

H.4 Stochastic Decompositions

As implied, for instance, by equation (OA15), each of the components of the welfare decomposition includes aggregate valuation considerations. Here, we formalize this insight by further decomposing i) the aggregate efficiency component into an expected aggregate efficiency component and an aggregate smoothing component, and ii) the redistribution component into an expected redistribution and a redistributive smoothing component. Similar decompositions can be constructed for intertemporal-sharing and risk-sharing.

Aggregate Efficiency. The aggregate efficiency component, Ξ^{AE}, can be decomposed into i) an expected aggregate efficiency component, Ξ^{EAE}, and ii) an aggregate smoothing component, Ξ^{AM}. Formally, at date t:

$$\Xi_t^{AE} = \underbrace{\mathbb{E}_{\pi_t(s^t)}\left[\omega_t\left(s^t\right)\right]\mathbb{E}_{\pi_t(s^t)}\left[\sum_i \frac{dV_t^{i|\lambda}\left(s^t\right)}{d\theta}\right]}_{=\Xi_t^{EAE}\text{ (Expected Aggregate Efficiency)}} + \underbrace{\mathbb{C}ov_{\pi_t(s^t)}\left[\frac{\omega_t\left(s^t\right)}{\pi_t\left(s^t\right)}, \sum_i \frac{dV_t^{i|\lambda}\left(s^t\right)}{d\theta}\right]}_{=\Xi_t^{AM}\text{ (Aggregate Smoothing)}},$$

where $\Xi^{AE} = \sum \omega_t \Xi_t^{AE}$. This decomposition is the standard asset pricing decomposition into the expected payoff and a risk compensation. The expected aggregate efficiency component, Ξ_t^{EAE}, captures the expected welfare gain across histories at a particular date. The aggregate smoothing component, Ξ_t^{AM}, captures whether aggregate efficiency gains take place in histories that a planner values more in aggregate terms. It should be evident that aggregate smoothing, based on the covariance of aggregate welfare gains across histories, is logically different from the risk-sharing and intertemporal-sharing components, Ξ^{RS} and Ξ^{IS}, based on cross-sectional covariances.

The welfare gains associated with eliminating aggregate business cycles in a representative-agent economy, (Lucas, 1987), arise from aggregate smoothing considerations. Finally, note it is possible to generate a similar decomposition that captures the part of aggregate efficiency welfare gains that are due to front-loading welfare gains, by using a covariance decomposition across dates.

Redistribution. Similarly to the aggregate efficiency component, the redistribution component Ξ^{RD} is shaped by valuation considerations, in this case, at the individual level. Here, we decompose Ξ^{RD} into i) an expected redistribution component, Ξ^{ER}, and a redistributive smoothing component, Ξ^{RM}. Formally, at date t:

$$\Xi_t^{RD} = \underbrace{\mathbb{C}ov_i\left[\omega^i, \sum_t \omega_t^i \mathbb{E}_{\pi_t(s^t)}\left[\omega_t\left(s^t\right)\right]\mathbb{E}_{\pi_t(s^t)}\left[\frac{dV_t^{i|\lambda}\left(s^t\right)}{d\theta}\right]\right]}_{=\Xi^{ER}\text{ (Expected Redistribution)}}$$
$$+ \underbrace{\mathbb{C}ov_i\left[\omega^i, \sum_t \omega_t^i \mathbb{C}ov_{\pi_t(s^t)}\left[\frac{\omega_t\left(s^t\right)}{\pi_t\left(s^t\right)}, \frac{dV_t^{i|\lambda}\left(s^t\right)}{d\theta}\right]\right]}_{=\Xi^{RM}\text{ (Redistributive Smoothing)}}.$$

This is, again, a standard asset pricing decomposition. The expected redistribution component, Ξ^{ER}, captures the welfare gains due to the expected welfare gains across histories at a particular date. When individuals with a high individual weight have higher expected welfare gains, a planner attributes this to the expected redistribution component. The redistributive smoothing component, Ξ^{RM}, captures whether individual welfare gains take place in histories that are more desirable for individuals with a higher individual weight. That is, the redistributive smoothing component will be non-zero for perturbations that smooth individual consumption for individuals with high individual weights.

H.5 Alternative Cross-Sectional Decompositions

Here, we provide two alternative cross-sectional decompositions of the risk-sharing and intertemporal-sharing components.

First, using equation (OA12), it is possible to decompose Ξ^{IS} into i) a raw intertemporal-sharing component, ii) a weight concentration component, and iii) a policy-weights coskewness component, as follows

$$\Xi^{IS} = \underbrace{\sum_t \omega_t \sum_{s^t} \omega_t\left(s^t\right) \mathbb{C}ov_i^{\Sigma}\left[\frac{\omega_t^i}{\omega_t}, \frac{dV_t^{i|\lambda}\left(s^t\right)}{d\theta}\right]}_{=\Xi^{RIS}\text{ (Raw Intertemporal-sharing)}}$$
$$+ \underbrace{\sum_t \omega_t \sum_{s^t} \omega_t\left(s^t\right) \mathbb{C}ov_i\left[\frac{\omega_t^i}{\omega_t}, \frac{\omega_t^i\left(s^t\right)}{\omega_t\left(s^t\right)}\right] \sum_i \frac{dV_t^{i|\lambda}\left(s^t\right)}{d\theta}}_{=\Xi^{WC}\text{ (Weight Concentration)}}$$
$$+ \underbrace{\sum_t \omega_t \sum_{s^t} \omega_t\left(s^t\right) \sum_i \left(\frac{\omega_t^i}{\omega_t} - 1\right)\left(\frac{\omega_t^i\left(s^t\right)}{\omega_t\left(s^t\right)} - 1\right)\left(\frac{dV_t^{i|\lambda}\left(s^t\right)}{d\theta} - \mathbb{E}_i\left[\frac{dV_t^{i|\lambda}\left(s^t\right)}{d\theta}\right]\right)}_{=\Xi^{PC}\text{ (Policy-weights Coskewness)}}.$$

The first component of Ξ^{IS}, Ξ^{RIS}, can be interpreted as an intertemporal-sharing component in which welfare gains at date t are not risk-discounted (i.e., raw). Note that the history s^t determinant of Ξ^{IS} relative to Ξ^{RIS} compare as follows

$$\mathbb{C}ov_i^{\Sigma}\left[\frac{\omega_t^i}{\omega_t}, \frac{\omega_t^i\left(s^t\right)}{\omega_t\left(s^t\right)}\frac{dV_t^{i|\lambda}\left(s^t\right)}{d\theta}\right] \quad \text{vs.} \quad \mathbb{C}ov_i^{\Sigma}\left[\frac{\omega_t^i}{\omega_t}, \frac{dV_t^{i|\lambda}\left(s^t\right)}{d\theta}\right],$$

where it is clear that intertemporal-sharing corrects welfare gains by risk through $\frac{\omega_t^i(s^t)}{\omega_t(s^t)}$, while raw intertemporal-sharing does not. Hence, the remaining two components, Ξ^{WC} and Ξ^{PC}, precisely capture the difference due to such risk correction.

The Ξ^{WC} component corrects for the fact that dynamic and stochastic weights are cross-sectionally correlated. Even though one may consider including Ξ^{WC} in the aggregate efficiency component, there are two good reasons not to do so. First, it would require knowledge of the cross-section of the dynamic and stochastic weights, which goes against expressing the aggregate efficiency component exclusively as a function of aggregate statistics. Second, $\Xi^{WC} = 0$ when markets are complete, which highlights that Ξ^{WC} relies on valuation differences across individuals.

The Ξ^{PC} component is based on the coskewness between dynamic and stochastic weights and the instantaneous welfare gain. Coskewness is a measure of how much three random variables jointly change. For instance, Ξ^{PC} could be non-zero even when $\mathbb{C}ov_i\left[\frac{\omega_t^i}{\omega_t}, \frac{\omega_t^i(s^t)}{\omega_t(s^t)}\right] = 0$. Also, coskewness is zero when the random variables are multivariate normal (Bohrnstedt and Goldberger, 1969), so it relies on higher-order moments.[23] Ξ^{WC} is also zero if one of ω_t^i, $\omega_t^i\left(s^t\right)$, or $\frac{dV_t^{i|\lambda}(s^t)}{d\theta}$ is constant across individuals.

I Extensions: Additional Results

In this section, we discuss additional results.

I.1 Pareto Problem

When introducing the efficiency/redistribution decomposition, we claim that those allocations that solve the Pareto problem, as defined in e.g. Ljungqvist and Sargent (2018) must feature a weakly negative efficiency component for any feasible perturbation given endowments and technologies. The Pareto problem consists of maximizing

$$\max_{\left\{c_t^i(s^t)\right\}} \sum_i \alpha^i V^i,$$

where V^i is defined as in (1), and where at each history it must be that

$$\sum_i c_t^i\left(s^t\right) = c_t\left(s^t\right),\ \forall t,\ \forall s^t,$$

where, given Inada conditions, it must be that $c_t^i\left(s^t\right) > 0$. In endowment economies, aggregate consumption $c_t\left(s^t\right)$ is predetermined, but more generally there could be more equations that determine how $c_t\left(s^t\right)$ is produced. For simplicity, here we assume that instantaneous utility exclusively depends on consumption, but it is straightforward to generalize the results.

Note that, at an optimum, it must be that

$$\alpha^i\left(\beta^i\right)^t \pi_t\left(s^t\right)\frac{\partial u_t^i\left(s^t\right)}{\partial c_t^i\left(s^t\right)} = \tilde{\eta}_t\left(s^t\right), \forall t,\ \forall s^t,$$

where $\tilde{\eta}_t\left(s^t\right)$ is the Lagrange multiplier in the resource constraint at history s^t. The optimality

[23] These terms are likely to be important in models that emphasize higher moments of the distribution of risks (e.g., Guvenen, Ozkan and Song (2014)).

conditions can in turn be written as

$$\omega^i \omega_t^i \omega_t^i \left(s^t\right) = \eta_t \left(s^t\right),$$

where $\eta_t(s^t)$ is a normalized version of $\tilde{\eta}_t(s^t)$. Note that

$$\sum_t \sum_{s^t} \omega^i \omega_t^i \omega_t^i \left(s^t\right) = \sum_t \sum_{s^t} \eta_t \left(s^t\right) \Rightarrow \omega^i = \sum_t \sum_{s^t} \eta_t \left(s^t\right),$$

which in turn implies that

$$\omega_t^i \omega_t^i \left(s^t\right) = \frac{\eta_t(s^t)}{\sum_t \sum_{s^t} \eta_t(s^t)} \Rightarrow \omega_t^i = \frac{\sum_{s^t} \eta_t(s^t)}{\sum_t \sum_{s^t} \eta_t(s^t)} \quad \text{and} \quad \omega_t^i \left(s^t\right) = \frac{\eta_t(s^t)}{\sum_{s^t} \eta_t(s^t)}.$$

These results imply that $\Xi^{RD} = \Xi^{IS} = \Xi^{RS} = 0$. If there exists a feasible perturbation (holding endowments and technologies fixed) of a particular allocation in which $\sum_t \sum_{s^t} \eta_t(s^t)\, dc_t(s^t) > 0$, this would increase the value of the objective, which immediately concludes perturbations to solutions to the Pareto problem must feature $\Xi^E \leq 0$.

I.2 Role of Transfers

Here, we explain how the ability to costlessly transfer resources across individuals impacts the welfare decomposition. Formally, if a DS-planner has access to a set of transfers $T_i^i(s^t)$ in units of the history welfare numeraire (here assumed to be consumption), so that individual budget constraints have the form

$$c_t^i \left(s^t\right) = T_i^i \left(s^t\right) + \ldots.$$

it follows immediately that the social value of such transfer equals the DS-weight of individual at that particular history:

$$\frac{dW^{DS}}{dT_i^i(s^t)} = \omega^i \omega_t^i \omega_t^i \left(s^t\right) = \tilde{\omega}_t^i \left(s^t\right).$$

When a planner can transfer resources costlessly across individuals, subject to $\sum_i T_i^i(s^t) = 0$, the availability of transfers endogenously restricts the variation of DS-weights across different individuals. For instance, a welfarist planner who can transfer resources freely across all individuals, at all dates and histories will equalize the DS-weights across all individuals, at all dates and histories. Given Proposition 5, this implies that this planner will only value aggregate efficiency. Similar conclusions can be reached when a DS-planner only has access to a subset of transfers.

I.3 Global Assessments

The body of the paper focuses on marginal welfare assessments because marginal welfare gains can be computed unambiguously — see e.g. Schlee (2013), which shows that consumer surplus, equivalent variation, and compensating variation are identical for marginal changes in a classical demand setup. However, it is important to understand how to make non-marginal welfare assessments.

Even for a single individual, there is no unambiguous approach to measure welfare gains or losses for non-marginal changes in meaningful units (money-metric) — see e.g., Silberberg (1972) or Mas-Colell, Whinston and Green (1995). This phenomenon is typically illustrated by the discrepancy between consumer surplus, equivalent variation, and compensating variation in classic demand theory. The same logic extends to aggregate welfare assessments and to the welfare decomposition. Despite this hurdle, it is possible to make judicious global welfare assessments.

In practice, it is possible to study global changes by parameterizing perturbations using a line integral, as illustrated in Section 4. Assuming that perturbations can be scaled by $\theta \in [0,1]$, where $\theta = 0$ corresponds to the status-quo and $\theta = 1$ corresponds to a global non-marginal change, it is possible to define a non-marginal welfare change as follows:

$$W^{DS}(\theta = 1) - W^{DS}(\theta = 0) = \int_0^1 \frac{dW^{DS}(\theta)}{d\theta} d\theta,$$

where θ is an explicit argument of $\frac{dW^{DS}(\theta)}{d\theta}$, defined as in (3) or (OA3). That is, by recomputing $\frac{dW^{DS}(\theta)}{d\theta}$ or $\frac{dW^{\lambda}(\theta)}{d\theta}$ along a particular path, it is possible to come up with a social welfare measure that is akin to consumer surplus, with the same logic applying to each of the components of the welfare decomposition. While using different paths will typically yield different global answers to the question of what are the gains from a global multidimensional perturbation, in practice it is often possible to find monotonic paths of integration, as defined by Zajac (1979) and Stahl (1984). In that case, there is no ambiguity on whether $\theta = 0$ is socially preferred to $\theta = 1$, or vice versa.

Two additional remarks are worth making. First, while the approach outlined here is the easiest to implement, it is possible to follow Alvarez and Jermann (2004) to consider global equivalent/compensating variation-like assessments for welfarist planners within the DS-weights framework. This will only be valid for aggregate assessments, not necessarily each of the components of the welfare decomposition.

Second, the potential for ambiguity of global assessments is not relevant if one is interested in using DS-planners to solve optimal policy problems, since $\frac{dW^{DS}}{d\theta}$ is unambiguously defined for any policy perturbation. Hence, if there is a point at which $\frac{dW^{DS}}{d\theta} = 0$ given the set of policy instruments, this will be a critical point and, under suitable second-order conditions, a local optimum. If there is a single local optimum and it is possible to establish that the optimum is interior, this optimum will be global. If there are multiple local optima, one could use the value of the social welfare function to rank them in the welfarist case. So welfarist planners can unambiguously rank any two policies globally. Outside of the welfarist case, one can look for monotonic paths of integration (Zajac, 1979; Stahl, 1984) to rank different local optima, so it is only when this is not possible to find such paths that there may be some global ambiguity when ranking two particular policies for DS-planners.[24] In general, one can choose a set of reasonable policy paths (e.g., linear paths or bounded paths) and compare the predictions for the associated welfare assessments both in aggregate and for each of the components of the welfare decomposition.

I.4 Inequality and Bounds

Concerns related to inequality often take a prominent role when assessing the welfare impact of policies. The welfare decomposition introduced in this paper highlights which particular forms of inequality matter for the determination of welfare assessments and their components.

Formally, by using the Cauchy-Schwarz inequality — which states that $\left|\mathbb{C}ov\left[x,y\right]\right| \leq \sqrt{\mathbb{V}ar\left[x\right]}\sqrt{\mathbb{V}ar\left[y\right]}$ — it is possible to provide bounds for Ξ^{RS}, Ξ^{IS}, and Ξ^{RD} based on the cross-sectional dispersion of normalized weights and the welfare gains, as follows:

$$\begin{aligned}
\left|\Xi^{RS}\right| &\leq \sum_{t}\omega^{t}\sum_{s^{t}}\mathbb{SD}_{i}^{\Sigma}\left[\omega_{t}^{i}\left(s^{t}\right)\right]\cdot\mathbb{SD}_{i}^{\Sigma}\left[\frac{dV_{t}^{i}\left(s^{t}\right)}{d\theta}\right]\\
\left|\Xi^{IS}\right| &\leq \sum_{t}\mathbb{SD}_{i}^{\Sigma}\left[\omega_{t}^{i}\right]\cdot\mathbb{SD}_{i}^{\Sigma}\left[\frac{dV_{t}^{i}}{d\theta}\right]\\
\left|\Xi^{RD}\right| &\leq \mathbb{SD}_{i}^{\Sigma}\left[\omega^{i}\right]\cdot\mathbb{SD}_{i}^{\Sigma}\left[\frac{dV^{i}}{d\theta}\right],
\end{aligned}$$

where $\mathbb{SD}_{i}^{\Sigma}\left[\cdot\right]$ denotes a cross-sectional standard deviation, where the variance is computed in sum form. This result shows that inequality considerations matter for the aggregate assessments of policies via the cross-sectional dispersion of normalized or the impact of a perturbation by itself. While cross-sectional standard deviations can bound the welfare effect of perturbations, the welfare decomposition is a function of covariances.

These bounds are helpful in practice because they can be computed using univariate statistics, i.e., cross-sectional standard deviations, and do not require the joint distribution of DS-weights and normalized consumption-equivalent effects, which are necessary to compute cross-sectional covariances (a multivariate statistic).

[24] Stahl (1984) proves that there always exist monotonic paths of integration in a classical demand context. While a formal proof of the existence of such paths for the general framework considered here is outside of the scope of this paper, there is no reason to believe this result cannot be extended to more general environments.

J Relation to Existing Work

J.1 Relation to Saez and Stantcheva (2016)

The notion of DS-planners introduced in Section E nests the generalized weight approach in Saez and Stantcheva (2016) and extends it to dynamic stochastic environments. Formally, while that paper considers welfare objectives that directly define the individual weight ω^i, DS-planners also define (potentially non-welfarist) dynamic and stochastic generalized weights, ω_t^i and $\omega_t^i(s^t)$, for each individual.[25]

Hence, DS-planners in i) static environments or ii) environments that exclusively feature generalized individual weights (but welfarist dynamic and stochastic weights) can be interpreted as special cases of the generalized weight approach in Saez and Stantcheva (2016). Although in that second case, our results use a different choice of lifetime welfare numeraire: Saez and Stantcheva (2016) naturally choose history consumption as their numeraire, but this choice is more subtle in dynamic environments, as explained in Section F. DS-planners that feature generalized dynamic and/or stochastic weights are not considered in that paper. It is also worth highlighting that their paper does not feature any form of welfare decomposition, even for static environments, so every result in Section 3 of this paper is unrelated to the results in Saez and Stantcheva (2016).

A central insight in Saez and Stantcheva (2016) is that by using (individual) generalized weights it is possible to accommodate alternatives to welfarism, such as equality of opportunity, libertarianism, or Rawlsianism, among others. Since our approach nests theirs, it can also accommodate these possibilities. There is scope to integrate these alternatives into dynamic stochastic environments.

J.2 Relation to Lucas (1987) and Alvarez and Jermann (2004)

It is common in papers that make welfare assessments in dynamic stochastic environments to compute welfare gains using consumption-equivalents, as in Lucas (1987), who measures the welfare gains associated with a policy change — specifically, the welfare gains associated with eliminating business cycles. Our approach, built using marginal arguments, connects directly to the results in Alvarez and Jermann (2004), who provide a marginal formulation of the approach in Lucas (1987). While the Lucas (1987) and Alvarez and Jermann (2004) approach is easily interpretable in representative agent economies, it has the pitfall that consumption-equivalents cannot be meaningfully aggregated when there are heterogeneous individuals. See, for instance, how Atkeson and Phelan (1994), Krusell and Smith (1999), or Krusell et al. (2009) carefully avoid aggregating consumption-equivalent welfare

[25] In general, unless they are based on a social welfare function, welfare assessments based on generalized individual weights (those considered in Saez and Stantcheva (2016)) are non-welfarist, yet they are Paretian and non-paternalistic. Welfare assessments based on generalized dynamic and stochastic weights (those considered in Section E of this paper) are non-welfarist, and typically non-Paretian and paternalistic.

gains across different individuals.

To illustrate these arguments, here we consider a perturbation for a given individual i, who could be a representative agent or not. We abstract from factor supply, for simplicity, and consider preferences of the form

$$V^i = \sum_t \left(\beta^i\right)^t \sum_{s^t} \pi_t\left(s^t\right) u^i\left(c_t^i\left(s^t\right)\right).$$

We suppose that the consumption of individual i at date t and history s^t can be written as

$$c_t^i\left(s^t\right) = (1-\theta)\,\underline{c_t^i}\left(s^t\right) + \theta\overline{c_t^i}\left(s^t\right),$$

where both $\underline{c_t^i}\left(s^t\right)$ and $\overline{c_t^i}\left(s^t\right)$ are sequences measurable with respect to history s^t. The sequence $\underline{c_t^i}\left(s^t\right)$ can be interpreted as a given initial consumption path — when $\theta = 0$ — and the sequence $\overline{c_t^i}\left(s^t\right)$ can be interpreted as a final consumption path — when $\theta = 1$. In the case of Lucas (1987), $\theta = 1$ corresponds to fully eliminating business cycles.

First, we compute the marginal gains from marginally reducing business cycles using a multiplicative consumption-equivalent, as in Lucas (1987) and Alvarez and Jermann (2004). Next, we compute the marginal gains using an additive consumption-equivalent.

Multiplicative Compensation. Lucas (1987) proposes using a time-invariant equivalent variation, expressed multiplicatively as a constant fraction of consumption at each date and history as follows

$$\sum_t \left(\beta^i\right)^t \sum_{s^t} \pi_t\left(s^t\right) u^i\left(\underline{c_t^i}\left(s^t\right)\left(1+\Lambda^i(\theta)\right)\right) = \sum_t \left(\beta^i\right)^t \sum_{s^t} \pi_t\left(s^t\right) u^i\left((1-\theta)\,\underline{c_t^i}\left(s^t\right) + \theta\overline{c_t^i}\left(s^t\right)\right), \tag{OA16}$$

where $\Lambda^i(\theta)$ implicitly defines the welfare gains associated with a policy indexed by θ. The exact definition in Lucas (1987) corresponds to solving for $\Lambda^i(\theta = 1)$.[26]

Following Alvarez and Jermann (2004), the derivative of the RHS of equation (OA16) is given by

$$\sum_t \left(\beta^i\right)^t \sum_{s^t} \pi_t\left(s^t\right) u^{i\prime}\left((1-\theta)\,\underline{c_t^i}\left(s^t\right) + \theta\overline{c_t^i}\left(s^t\right)\right) \frac{dc_t^i\left(s^t\right)}{d\theta}, \tag{OA17}$$

where $\frac{dc_t^i\left(s^t\right)}{d\theta} = \overline{c_t^i}\left(s^t\right) - \underline{c_t^i}\left(s^t\right)$.

[26] Alternatively, one could define a compensating variation as

$$\sum_t \left(\beta^i\right)^t \sum_{s^t} \pi_t\left(s^t\right) u^i\left(\underline{c_t^i}\left(s^t\right)\right) = \sum_t \left(\beta^i\right)^t \sum_{s^t} \pi_t\left(s^t\right) u^i\left(\left((1-\theta)\,\underline{c_t^i}\left(s^t\right) + \theta\overline{c_t^i}\left(s^t\right)\right)\left(1+\Lambda^i(\theta)\right)\right).$$

Analogously, the derivative of the LHS of equation (OA16) is given by

$$\sum_t \left(\beta^i\right)^t \sum_{s^t} \pi_t \left(s^t\right) u^{i\prime} \left(\underline{c_t^i} \left(s^t\right) \left(1 + \Lambda^i (\theta)\right)\right) \underline{c_t^i} \left(s^t\right) \frac{d\Lambda^i}{d\theta}. \tag{OA18}$$

Hence, combining (OA17) and (OA18) and solving for $\frac{d\Lambda^i}{d\theta}$, yields the marginal cost of business cycles, as defined in Alvarez and Jermann (2004). Formally, we can express $\frac{d\Lambda^i}{d\theta}$ as

$$\frac{d\Lambda^i}{d\theta} = \frac{\sum_t \left(\beta^i\right)^t \sum_{s^t} \pi_t \left(s^t\right) u^{i\prime} \left((1-\theta)\, \underline{c_t^i} \left(s^t\right) + \theta \overline{c_t^i} \left(s^t\right)\right) \frac{dc_t^i\left(s^t\right)}{d\theta}}{\sum_t \left(\beta^i\right)^t \sum_{s^t} \pi_t \left(s^t\right) u^{i\prime} \left(\underline{c_t^i} \left(s^t\right) \left(1 + \Lambda^i (\theta)\right)\right) \underline{c_t^i} \left(s^t\right)} = \sum_t \sum_{s^t} \tilde{\omega}_t^i \left(s^t\right) \frac{dc_t^i \left(s^t\right)}{d\theta}, \tag{OA19}$$

where DS-weights are given by

$$\tilde{\omega}_t^i \left(s^t\right) = \frac{\left(\beta^i\right)^t \pi_t \left(s^t\right) u^{i\prime} \left((1-\theta)\, \underline{c_t^i} \left(s^t\right) + \theta \overline{c_t^i} \left(s^t\right)\right)}{\sum_t \left(\beta^i\right)^t \sum_{s^t} \pi_t \left(s^t\right) u^{i\prime} \left(c_t^i \left(s^t\right) \left(1 + \Lambda^i (\theta)\right)\right) \underline{c_t^i} \left(s^t\right)}. \tag{OA20}$$

Additive Compensation. Here, we would like to contrast the approach in Lucas (1987) to one that relies on a time-invariant equivalent variation, expressed additively in terms of consumption at each date and history as follows:

$$\sum_t \left(\beta^i\right)^t \sum_{s^t} \pi_t \left(s^t\right) u_i \left(\underline{c_t^i} \left(s^t\right) + \mathcal{A}^i (\theta)\right) = \sum_t \left(\beta^i\right)^t \sum_{s^t} \pi_t \left(s^t\right) u_i \left((1-\theta)\, \underline{c_t^i} \left(s^t\right) + \theta \overline{c_t^i} \left(s^t\right)\right).$$

Following the same steps as above to find the counterpart of equation (OA19), we find that

$$\frac{d\mathcal{A}^i}{d\theta} = \frac{\sum_t \left(\beta^i\right)^t \sum_{s^t} \pi_t \left(s^t\right) u^{i\prime} \left((1-\theta)\, \underline{c_t^i} \left(s^t\right) + \theta \overline{c_t^i} \left(s^t\right)\right) \frac{dc_t^i\left(s^t\right)}{d\theta}}{\sum_t \left(\beta^i\right)^t \sum_{s^t} \pi_t \left(s^t\right) u^{i\prime} \left(\underline{c_t^i} \left(s^t\right) + \mathcal{A}^i (\theta)\right) \underline{c_t^i} \left(s^t\right)} = \sum_t \sum_{s^t} \tilde{\omega}_t^i \left(s^t\right) \frac{dc_t^i \left(s^t\right)}{d\theta}, \tag{OA21}$$

where DS-weights are given by

$$\tilde{\omega}_t^i \left(s^t\right) = \frac{\left(\beta^i\right)^t \pi_t \left(s^t\right) u^{i\prime} \left((1-\theta)\, \underline{c_t^i} \left(s^t\right) + \theta \overline{c_t^i} \left(s^t\right)\right)}{\sum_t \left(\beta^i\right)^t \sum_{s^t} \pi_t \left(s^t\right) u^{i\prime} \left(\underline{c_t^i} \left(s^t\right) + \mathcal{A}^i (\theta)\right)}. \tag{OA22}$$

Comparison and Implications. We focus on comparing equations (OA19) and (OA21) when $\theta = 0$ — similar insights emerge when $\theta \neq 0$. When $\theta = 0$, equations (OA20) and (OA22) become

$$\tilde{\omega}_t^i\left(s^t\right) = \frac{\left(\beta^i\right)^t \pi_t\left(s^t\right) u^{i\prime}\left(\underline{c}_t^i\left(s^t\right)\right)}{\sum_t \left(\beta^i\right)^t \sum_{s^t} \pi_t\left(s^t\right) u^{i\prime}\left(\underline{c}_t^i\left(s^t\right)\right) \underline{c}_t^i\left(s^t\right)} \quad \text{(multiplicative)} \tag{OA23}$$

$$\tilde{\omega}_t^i\left(s^t\right) = \frac{\left(\beta^i\right)^t \pi_t\left(s^t\right) u^{i\prime}\left(\underline{c}_t^i\left(s^t\right)\right)}{\sum_t \left(\beta^i\right)^t \sum_{s^t} \pi_t\left(s^t\right) u^{i\prime}\left(\underline{c}_t^i\left(s^t\right)\right)}. \quad \text{(additive)} \tag{OA24}$$

Two major insights emerge from equations (OA23) and (OA24). First, the DS-weights defined for the additive case in equation (OA24) exactly correspond to the product of the normalized dynamic and stochastic weights for a welfarist planner, as defined in (8) and (9). Second, the denominator of the DS-weights in the multiplicative case is multiplied by $\underline{c}_t^i\left(s^t\right)$ at all dates and histories. This captures the fact that the welfare assessment is computed as a fraction of consumption at each date and history, not in units of the consumption good. The presence of $\underline{c}_t^i\left(s^t\right)$ in the denominator is what complicates the aggregation of welfare assessments using the Lucas (1987) approach.

While both Lucas (1987) and Alvarez and Jermann (2004) study representative-agent environments, others have used a similar approach in environments with heterogeneity; see e.g., Atkeson and Phelan (1994), Krusell and Smith (1999), or Krusell et al. (2009), among many others. However, as highlighted by these papers, a well-known downside of the Lucas (1987) approach is that it does not aggregate meaningfully because individual welfare assessments are reported as constant shares of individual consumption. Our approach, implicitly based on an additive compensation, allows for meaningful aggregation among heterogeneous individuals.

Relation to EV, CV, and CS. Finally, note that the analysis in this section illustrates how the marginal approach relates to the conventional approaches in classic demand theory: equivalent variation (EV), compensating variation (CV), and consumer surplus (CS). The approach of Lucas (1987) and Alvarez and Jermann (2004), and the alternative version described in Footnote 26 are the dynamic counterpart of compensating and equivalent variations, expressed in proportional terms, in a dynamic stochastic environment. Hence, the analysis of this section shows that a DS-planner can be used to operationalize the counterpart of all three notions — either proportionally or additively — in dynamic stochastic environments. As expected, these considerations only matter away from the $\theta = 0$ case. However, the consumer surplus approach yields the most straightforward approach to making global assessments, as explained in Section I.3.

J.3 Relation to Existing Welfare Decompositions

Our paper is not the first to introduce a decomposition of welfare assessments in different components. In fact, most of the existing literature that applies welfare decompositions to specific environments follows versions of the decompositions introduced by Benabou (2002) and Floden (2001). There is also the more recent decomposition introduced by Bhandari et al. (2021). We discuss how our approach is related to both of these next.

Benabou (2002)/Floden (2001). The Benabou (2002)/Floden (2001) approach is based on first computing certainty-equivalent consumption levels for individuals and then building measures of inequality from the distribution of such certainty-equivalents. The starting point for the Benabou (2002) approach is the (incorrect) presumption that the welfarist approach cannot distinguish the effects of policy that operate via efficiency, missing markets, and redistribution. Benabou (2002) explicitly writes:

> "*I will also compute more standard social welfare functions, which are aggregates of (intertemporal) utilities rather than risk-adjusted consumptions. These have the clearly desirable property that maximizing such a criterion ensures Pareto efficiency. On the other hand, it will be seen that they cannot distinguish between the effects of policy that operate through its role as a substitute for missing markets, and those that reflect an implicit equity concern.*"

In this paper, we have shown that it is possible to distinguish — using standard social welfare functions — the effects of policy that operate through efficiency, including in economies with missing markets, and redistribution/equity. As Benabou (2002) points out, his approach may conclude that Pareto-improving policies are undesirable: this can never occur for welfarist planners, as explained in Section 3. It is only when considering non-welfarist planners — such as some DS-planners introduced in Section E — that perturbations that individuals find Pareto-improving are undesirable for a particular DS-planner. In those cases, our welfare decomposition is precise in the way in which such departures take place.

In terms of properties, it is evident that the Benabou (2002)/Floden (2001) approach does not satisfy Proposition 4a), in which we show that welfarist planners conclude that the risk-sharing and intertemporal-sharing components are zero when markets are complete; Proposition 4b), in which we show that welfarist planners conclude that intertemporal-sharing component is zero when individuals can freely trade a riskless bond; and Proposition 2a), in which we show that different welfarist planners exclusively disagree on the redistribution component, among others. The Benabou (2002)/Floden (2001) approach is only invariant to preference-preserving transformations because it is exclusively defined for environments in which all individuals have identical utility functions.

Bhandari et al. (2021). The decomposition introduced by Bhandari et al. (2021) considers a utilitarian planner with arbitrary Pareto weights α_i, although it seems obvious to apply to general social welfare functions. In contrast to Benabou (2002)/Floden (2001), the approach of Bhandari et al. (2021) is defined for general dynamic stochastic economies in which individuals may have different preferences.

For simplicity, we consider a scenario in which there is a single consumption good. In this environment, Bhandari et al. (2021) propose to first decompose the consumption of a given individual at a given date and history as

$$c_t^i\left(s^t\right) = C \times w_i \times \left(1 + \varepsilon_t^i\left(s^t\right)\right), \tag{OA25}$$

where C captures aggregate lifetime consumption, w_i captures the share of individual i's consumption relative to the aggregate and $1 + \varepsilon_t^i\left(s^t\right)$ captures any residual variation. While equation (OA25) may resemble the triple of individual, dynamics, and stochastic weights introduced in Lemma 1, it is conceptually different. In particular, the decomposition in equation (OA25) decomposes consumption, $c_t^i\left(s^t\right)$, while our weights decompose social marginal valuations. Only heuristically, the term w_i in (OA25) can be mapped to our normalized individual weight, while $1 + \varepsilon_t^i\left(s^t\right)$ can be mapped to both dynamic and stochastic weights.

Bhandari et al. (2021) then introduce a second-order Taylor expansion around a midpoint to write welfare differences (partially adopting the notation in that paper) as follows:

$$\mathcal{W}^B - \mathcal{W}^A \simeq \underbrace{\int \phi_i \Gamma di}_{\text{agg. efficiency}} + \underbrace{\int \phi_i \Delta_i di}_{\text{redistribution}} + \underbrace{\int \phi_i \gamma_i \Lambda_i di}_{\text{insurance}}, \tag{OA26}$$

where $\phi_i = \alpha_i \sum_t \sum_{s^t} \frac{\partial u_i\left(s^t\right)}{\partial c_t^i} c_t^i\left(s^t\right)$ denotes quasi-weights — using the terminology in Bhandari et al. (2021) — and γ_i is a measure of risk-aversion, $-c_t^i\left(s^t\right) \frac{\partial^2 u_i\left(s^t\right)}{\partial\left(c_t^i\right)^2} / \frac{\partial u_i\left(s^t\right)}{\partial c_t^i}$, and where $\Gamma = \ln C^B - \ln C^A$, $\Delta_i = \ln w_i^B - \ln w_i^A$, and $\Lambda_i = -\frac{1}{2}\left[\mathbb{V}ar_i\left[\ln c_i^B\right] - \mathbb{V}ar_i\left[\ln c_i^A\right]\right]$. That paper decomposes $\mathcal{W}^B - \mathcal{W}^A$ into three terms as follows:

$$1 = \underbrace{\frac{\int \phi_i \Gamma di}{\mathcal{W}^B - \mathcal{W}^A}}_{\text{agg. efficiency}} + \underbrace{\frac{\int \phi_i \Delta_i di}{\mathcal{W}^B - \mathcal{W}^A}}_{\text{redistribution}} + \underbrace{\frac{\int \phi_i \gamma_i \Lambda_i di}{\mathcal{W}^B - \mathcal{W}^A}}_{\text{insurance}}. \tag{OA27}$$

Bhandari et al. (2021) establish three properties of the decomposition in equation (OA27): a) a welfare change that affects aggregate consumption C but not $\{w_i, \varepsilon_i\}_i$ is exclusively attributed to aggregate efficiency; b) a welfare change that affects expected shares $\{w_i\}_i$ but not C and $\{\varepsilon_i\}_i$ is exclusively attributed to redistribution; c) a welfare change that affects $\{\varepsilon_i\}_i$ but not C and

$\{w_i\}_i$ is exclusively attributed to insurance. The insurance component in Bhandari et al. (2021) is heuristically related to the risk-sharing and intertemporal-sharing components in our paper. Bhandari et al. (2021) also establish a fourth property, reflexivity, which our approach also satisfies. These properties are conceptually the counterpart of Proposition 5a), since they consider properties of a decomposition for particular perturbations. However, it should be evident that properties a), b), and c) in Bhandari et al. (2021) neither imply nor are implied by the properties that we establish in Proposition 5a). This occurs because properties a), b), and c) consider proportional changes while Proposition 5a) considers changes in levels, with both the proportional and level approaches being different but reasonable.[27]

The decomposition of Bhandari et al. (2021) does not have a counterpart to Propositions 4 and 5. That is, it is possible to consider complete market economies in which the decomposition of Bhandari et al. (2021) attributes welfare changes to their insurance component. More importantly, it follows from (OA27) that changing the Pareto weights α_i that a utilitarian planner assigns to an individual or simply multiplying the lifetime utility of a single individual by a constant factor — a preference-preserving transformation that has no impact on allocations — will change all three elements (aggregate efficiency, redistribution, insurance) of the decomposition introduced by Bhandari et al. (2021). Formally, it follows from the definition of ϕ_i above that a change in α_i or a linear transformation of utilities will change ϕ_i and consequently each of the three elements on the right-hand side of equation (OA26). Critically, $\mathcal{W}^B - \mathcal{W}^A$ in equation (OA26) (as well as ϕ_i) is expressed in utils, not consumption units or any other common numeraire.[28] Hence, changes in Pareto weights or utility transformations directly affect all the components of the decomposition, including aggregate efficiency and insurance in equation (OA27).

[27]Note that by writing $c_t^i\left(s^t\right) = C \times w_i \times \left(1+\varepsilon_t^i\left(s^t\right)\right)$, we can express $\frac{dc_t^i(s^t)}{d\theta}$ as follows:

$$\frac{dc_t^i\left(s^t\right)}{d\theta} = \frac{dC}{d\theta} \times w_i \times \left(1+\varepsilon_t^i\left(s^t\right)\right) + C \times \frac{dw_i}{d\theta} \times \left(1+\varepsilon_t^i\left(s^t\right)\right) + C \times w_i \times \frac{d\left(1+\varepsilon_t^i\left(s^t\right)\right)}{d\theta}.$$

In this case, even when $\frac{dw_i}{d\theta} = \frac{d\left(1+\varepsilon_t^i\left(s^t\right)\right)}{d\theta} = 0$, a change in $\frac{dC}{d\theta}$, by virtue of being *proportional* to existing consumption, does not imply a uniform change in $\frac{dc_t^i(s^t)}{d\theta}$ across individuals, dates, and histories. A similar logic applies to changes in $\frac{dw_i}{d\theta}$ and $\frac{d\varepsilon_t^i(s^t)}{d\theta}$. More generally, the decompositions yield different conclusions. For instance, the decomposition in Bhandari et al. (2021) attributes welfare gains associated to smoothing business cycles in a representative agent economy — as in Lucas (1987) — to insurance, while our decomposition attributes such gains to the aggregate insurance subcomponent of aggregate efficiency.

[28]Bhandari et al. (2021) explain how $\mathcal{W}^B - \mathcal{W}^A$ is measured in utils as follows:

> *"Quasi-weights $\{\phi_i\}_i$ convert percent changes $\{\Gamma, \Delta_i, \Lambda_i\}_i$ that into a welfare change $\mathcal{W}^B - \mathcal{W}^A$, measured in utils."*