A Dynamic Theory of Optimal Tariffs

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May 2025

Abstract

The classic tariff formula states that the optimal unilateral tariff equals the inverse of the foreign export supply elasticity. We generalize this result and show that an *intertemporal tariff formula* characterizes the efficient tariff in a large class of dynamic heterogeneous agent (HA) economies with multiple goods. Intertemporal export supply elasticities and relative tariff revenue weights are sufficient statistics for the optimal tariff that decentralizes the efficient allocation. We also develop a general theory of second-best optimal tariffs. In dynamic HA incomplete markets economies, Ramsey optimal tariffs trade off intertemporal terms of trade manipulation against production efficiency, risk-sharing, and redistribution. Intertemporal export supply elasticities and relative tariff revenue weights remain sufficient statistics for the intertemporal terms of trade manipulation motive of second-best optimal tariffs. We apply our results to a quantitative heterogeneous agent New Keynesian (HANK) model with trade.

JEL codes: D31, D52, D60, E21, E52, F13, F41, H21

Keywords: intertemporal tariff formula, intertemporal export supply elasticities, heterogeneous agents, sequence-space Jacobians

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We are grateful to Sam Kortum for helpful conversations.

1 Introduction

What is the optimal tariff? The classic answer to this question in a two-country environment is that Home's optimal tariff τ_t at date t is equal to one over Foreign's export supply elasticity ε_t,

$$\tau_t = \frac{1}{\varepsilon_t}. \tag{1}$$

This formula emerges naturally when Foreign produces a single tradable good and Foreign's export supply x_t^* is a static function of the import price, $x_t^* = \mathcal{X}_t^*(p_t)$. When choosing the optimal level of imports, the Home planner trades off the utility benefit of consumption against two costs. A unit of imports has a direct monetary cost of p_t. But demanding more imports also leads to an increase in the price, which is required in equilibrium to incentivize Foreign to supply more exports. And this equilibrium price response is proportional to the inverse of the static export supply elasticity $\varepsilon_t = \frac{\partial \log \mathcal{X}_t^*}{\partial \log p_t}$. Unlike the planner, households in Home do not internalize this price impact when making consumption decisions. The optimal import tariff incentivizes households to demand the socially optimal level of imports.

This paper develops a theory of optimal tariffs in dynamic environments where Foreign's export supply is an intertemporal rather than a static function of prices, $x_t^* = \mathcal{X}_t^*(\boldsymbol{p})$, with $\boldsymbol{p} = \{p_k\}_{k\geq 0}$. Foreign's export supply may respond to price changes both at earlier dates due to anticipation effects or at later dates due to endogenous persistence. As emphasized by Auclert et al. (2024b), static response functions—such as the one underlying the classic tariff formula—cannot be microfounded in modern models featuring intertemporal budget constraints and forward-looking behavior. We show in our applications that important benchmark models of intertemporal trade can be represented in terms of the intertemporal export supply function $\mathcal{X}^*(\boldsymbol{p})$.

Intertemporal tariff formula. Our first main result is an intertemporal tariff formula that generalizes the classic one to dynamic heterogeneous agent models of intertemporal trade. It takes the form

$$\tau_t = \frac{1}{\boldsymbol{\omega}_t' \boldsymbol{\mathcal{E}}_t}. \tag{2}$$

We denote by $\boldsymbol{\mathcal{E}}$ the sequence-space Jacobian (Auclert et al., 2021) of the intertemporal export supply function $\mathcal{X}^*$, that is, the infinite matrix of partial log derivatives $\frac{\partial \log \mathcal{X}_k^*}{\partial \log p_t}$. The formula also features relative tariff revenue weights, with $\boldsymbol{\omega}$ denoting the infinite matrix with entries $\omega_{kt} = \frac{\tau_k p_k x_k^*}{\tau_t p_t x_t^*}$. We denote by $\boldsymbol{\mathcal{E}}_t$ and $\boldsymbol{\omega}_t'$ the t-th column and row of the respective matrices, so the intertemporal tariff formula can also be written as $\frac{1}{\tau_t} = \sum_k \omega_{kt} \frac{\partial \log \mathcal{X}_k^*}{\partial \log p_t}$.

While the contemporaneous export elasticity ε_t (*XE*) is a sufficient statistic for the optimal tariff according to the classic formula (1), our intertemporal tariff formula (2) identifies intertemporal export elasticities $\boldsymbol{\mathcal{E}}$ (*iXEs*), as the relevant determinants of optimal tariffs in dynamic models of intertemporal trade. Together with relative tariff revenue weights, they are sufficient statistics for the optimal tariff. The economic intuition for equation (2) is similar to that of the classic formula (1). The Home planner internalizes that demanding more imports from Foreign at date t leads to a rise in the price of imports in equilibrium. Unlike in the classic case, however, intertemporal linkages imply that this demand impulse at date t shows up in market clearing conditions at all dates k, exerting pressure on prices in proportion to the off-diagonal entries of the iXE matrix $\boldsymbol{\mathcal{E}}$, which can be interpreted as intertemporal cross-price elasticities. Therefore, the Home planner internalizes that changing import demand at date t affects the entire sequence of prices $\boldsymbol{p}$ in equilibrium, thus changing the cost of inframarginal trade at all dates. To make households internalize this price impact—both contemporaneous and intertemporal price impact—the required import tariff takes the form (2).

Second-best tariffs. The intertemporal tariff formula (2) determines the optimal tariff whenever the Home allocation is efficient internally. It therefore characterizes the tariff that helps decentralize the efficient allocation—possibly in conjunction with other instruments. In this case, the planner uses tariffs for a single purpose: to manipulate Home's intertemporal terms of trade. Whenever the efficient allocation is not attainable, the planner may find it desirable to use tariffs to also target other inefficiencies.

We develop a general theory of second-best optimal tariffs that complements our intertemporal tariff formula. Our second main result is a Ramsey targeting rule that identifies the normative considerations relevant for second-best tariffs. In a large class of dynamic heterogeneous agent models, the optimal second-best tariff trades off intertemporal terms of trade manipulation against gains from production efficiency, risk-sharing, and redistribution. Nonetheless, we show that intertemporal export supply elasticities $\boldsymbol{\mathcal{E}}$, together with revenue weights, remain a sufficient statistic for the terms of trade manipulation motive of second-best tariffs.

Micro-foundations of intertemporal export supply elasticities. The classic tariff formula (1) corresponds to the case where the iXE matrix $\boldsymbol{\mathcal{E}}$ is diagonal, with $\boldsymbol{\mathcal{E}}_{kt} = \frac{\partial \log \mathcal{X}_k^*}{\partial \log p_t} = 0$ for all $k \neq t$. This is the case where intertemporal cross-price elasticities are zero. We explore the shape and importance of the off-diagonal entries of $\boldsymbol{\mathcal{E}}$ across three benchmark models of intertemporal trade that provide different micro-foundations for the intertemporal export supply function $\mathcal{X}^*$. We begin by studying a representative agent (RA) endowment economy in Section 3.1, which allows us to revisit and relate our results to the seminal analysis of Costinot et al. (2014). We show that $\boldsymbol{\mathcal{E}}$ has a stark shape in environments with permanent-income consumers, featuring a positive diagonal and negative and *flat* off-diagonal entries. Permanent-income consumers are infinitely forward-looking and consume the annuity value of the change in lifetime wealth following an anticipated price change at date t. Their consumption response is therefore the same across all dates $k \neq t$. This implies that all elements of the t-th column of $\boldsymbol{\mathcal{E}}$ are identical, except for the diagonal corresponding to the contemporaneous response. Our next result shows that while $\boldsymbol{\mathcal{E}}$ is not diagonal in this setting, its off-diagonal terms exactly cancel out in the determination of the optimal tariff. Tariffs are therefore determined exclusively by the diagonal of $\boldsymbol{\mathcal{E}}$, just *as if* the classic tariff formula applied.

Models with permanent-income consumers come deceptively close to replicating the classic tariff formula. While the iXE matrix $\boldsymbol{\mathcal{E}}$ is not diagonal, the off-diagonal intertemporal cross-price elasticities exactly cancel out in the determination of the optimal tariff. We next show that this coincidence breaks down in heterogeneous agent (HA) models with incomplete markets. Crucially, intertemporal cross-price elasticities are no longer constant when markets are incomplete. Consumers are no longer infinitely forward-looking. Instead, their effective planning horizons depend on their proximity to the borrowing constraint. For dates sufficiently far in advance of the anticipated price change, Foreign export supply does not respond at all. Shortly before and after the price change, however, the export supply response is much stronger than in the RA benchmark. In the presence of incomplete markets, households have larger marginal propensities to consume (MPC) out of near-contemporaneous windfalls as a result.

Finally, we present a micro-foundation of $\mathcal{X}^*$ in a New Keynesian model in Appendix C in the spirit of Farhi and Werning (2017). In both previous applications, the export supply function $\mathcal{X}^*$ was a strictly partial equilibrium (PE) object. But our theory also applies to richer environments where $\mathcal{X}^*$ features general equilibrium (GE) dynamics that are *internal* to Foreign. What is key is that $\mathcal{X}^*$ never accounts for GE adjustments that work through international trade. In the New Keynesian model, we show that $\mathcal{X}^*$ also accounts for business cycle dynamics that are internal to Foreign, something that Home exploits by adjusting the optimal import tariff.

Optimal tariffs according to a quantitative HANK model with trade. We leverage our analytical results to study optimal tariffs in a quantitative heterogeneous agent New Keynesian (HANK) model with trade, extending the canonical open economy HANK model developed by Auclert et al. (2024a) to the case of two large countries. There are J goods that come in differentiated Home- and Foreign-produced varieties. Firms face adjustment costs when changing prices, giving rise to sectoral New Keynesian Phillips curves. Households have homothetic CES preferences over the $2 \times J$ goods. In the tradition of the Bewley-Huggett-Aiyagari incomplete markets model, households face uninsurable income risk and are subject to a borrowing limit. Our calibrated quantitative model matches both trade elasticities and key moments of the income and wealth distribution.

Our quantitative analysis starts with a positive expoloration of the consequences of tariff shocks. In response to a permanent 10% increase in import tariffs, Home consumption rises by 0.5% in the long run, while labor supply and output fall by roughly 1%. Exports fall by nearly 20% while imports only contract by 15%, leading to a long-run worsening of the trade balance. In response to a temporary import tariff shock, the trade balance does improve for some time and Home production rises on impact.

We present the Ramsey problem for optimal second-best tariff policy in Appendix D and derive the optimality conditions that characterize the Ramsey plan. We compute the Ramsey steady state (RSS) and show that it implies an optimal long-run import tariff of 5.4%. We then compute the optimal tariff dynamics in response to a TFP shock from a timeless perspective. As Home productivity rises by 1%, the optimal import tariff also increases by 0.5%, leading to a moderate improvement in Home's trade balance.

Related literature. Our paper builds on the seminal work of Costinot et al. (2014). There is a vast literature studying optimal tariffs but much of this work is cast in static environments.[1] Costinot et al. (2014) was the first paper to develop a theory of optimal tariffs in dynamic environments. Our dual approach is complementary to their primal approach and identifies intertemporal export supply elasticities as a key determinant of optimal tariffs in both first-best and second-best environments. We also extend their analysis to HA models and show that the iXE matrix $\boldsymbol{\mathcal{E}}$ takes a starkly different shape in the presence of incomplete markets. As a result, optimal tariffs are no longer determined *as if* by the classic tariff formula, unlike in environments with permanent-income consumers.

Other papers in the trade literature that have influenced our work include Costinot et al. (2015), who characterize optimal tariffs in a static model of Ricardian comparative advantage, and Lashkaripour (2021), who derives a related sufficient statistics formula in a static multi-good environment. Costinot and Werning (2023) study optimal tariff policy in a static environment with redistribution concerns. Our results build on theirs by showing that a related but distinct risk-sharing motive emerges in dynamic incomplete market environments where households are exposed to uninsurable risk.

Our paper is more broadly related to prior work that characterizes constrained efficiency and optimal second-best policies in heterogeneous agent environments. Farhi and Werning (2016) develop a general theory of financial transaction taxes in an environment with heterogeneous households, a general supply side, and a flexible representation of nominal rigidities. Our approach builds closely on theirs, especially our characterization of the second-best tariff formula. We cast our result in terms of wedges, as they do, which succinctly summarize sources of inefficiency relative to the first-best allocation. Our main point of departure from their analysis is that we study a two-country model where the planning problem only puts welfare weights on households in one country. This gives rise to a new intertemporal terms of trade manipulation motive that a global planner would not act on. Relatedly, Farhi and Werning (2017) show in an open-economy setting that aggregate demand externalities make even the complete markets competitive equilibrium constrained inefficient, which motivates the use of fiscal transfers.

Our paper also contributes to the quickly growing literature in macroeconomics that leverages sequence-space methods to study heterogeneous agent models. The seminal work by Auclert

[1] See, among many others, Johnson (1953).

et al. (2021) introduced and popularized sequence-space Jacobians. Auclert et al. (2024b) derive an intertemporal Keynesian cross that generalizes the static one to HA environments that admit a sequence-space representation in terms of an intertemporal consumption function. We show that an intertemporal tariff formula generalizes the static one to HA environments that admit a sequence-space representation for the rest of the world in terms of an intertemporal export supply function. Our paper also builds on Dávila and Schaab (2023a), who extend this sequence-space approach to optimal policy problems and welfare analysis in HA economies. We extend their methods for solving the primal and dual Ramsey problems to two-country HA environments where intertemporal terms of trade manipulation emerges as a new motive of optimal policy. We illustrate the power of these tools in a quantitative HANK model that extends the canonical open-economy HANK model of Auclert et al. (2024a) to a setting with two large countries.

Finally, there is a new and quickly growing strand of literature in response to current tariff policy debate. Many of these papers address positive questions. Auclert et al. (2025) show that tariff shocks are contractionary in a representative agent New Keynesian model. They argue that consideration for the short-run recessionary effects of import tariffs lower the optimal long-run tariff. Costinot and Werning (2025) show that tariffs may have long-lasting effects on the trade balance and reduce trade deficits if the Engel curves for aggregate imports and exports are convex. Itskhoki and Mukhin (2025a) show that valuation effects are a key channel through which tariff policy affects the trade balance. Aguiar et al. (2025) characterize when a trade war that imposes balanced trade can be consistent with given initial net foreign asset positions. Caliendo et al. (2025) study the determinants of endogenous trade imbalances across countries in a dynamic complete markets model. Rodríguez-Clare et al. (2025) use a dynamic trade and reallocation model with downward nominal wage rigidity to show that recent U.S. tariff policy would expand manufacturing employment but at the cost of declines in service and agricultural employment. Ignatenko et al. (2025) study how the welfare implications of recent U.S. tariff policy depend on whether and to what extent trading partners relatiate. Itskhoki and Mukhin (2025b) characterize optimal tariffs under policy objectives that go beyond social welfare—such as maximizing revenue, closing the trade deficit, or increasing manufacturing employment. They focus on representative agent environments where long-run bilateral and aggregate trade deficits reflect differential returns on assets and liabilities, and show that the planner trades off terms of trade manipulation against the cost of negative valuation effects. Finally, several papers study the interaction between tariffs and optimal monetary policy, including Bianchi and Coulibaly (2025), who show that the optimal monetary policy response to a tariff shock is expansionary, and Werning et al. (2025), who show that tariffs appear as cost-push shocks in a standard New Keynesian model and that optimal monetary policy partially accommodates the shock and allows for higher short-run inflation.

2 The Intertemporal Tariff Formula

2.1 The Classic Tariff Formula

Consider a world economy with two large countries, Home and Foreign. Time is discrete and indexed by $t \in \{0, 1, \ldots\}$. A representative household in Home consumes a basket of domestically produced goods, c_t, and a tradable good produced by and imported from Foreign, m_t. The household's consumption preferences are $\sum_t \beta^t u(c_t, m_t)$. To derive the classic tariff formula, we assume that Home faces a *static* budget constraint, $y_t = c_t + p_t m_t$; total consumption expenditures must equal a given per-period level of income y_t. We denote by p_t the price of the imported good relative to the basket of Home goods. Finally, equilibrium requires that Home imports are equal to Foreign exports, $m_t = x_t^*$.

The key assumption underpinning the classic tariff formula is that there exists a *static* export supply function $\mathcal{X}_t^*$ that is differentiable and determines the level of Foreign export supply at date t as a function of the contemporaneous price,

$$x_t^* = \mathcal{X}_t^*(p_t). \tag{3}$$

The planning problem of Home is therefore to choose an allocation $\{c_t, m_t\}$ and prices $\{p_t\}$ subject to the domestic budget constraint and the implementability condition $m_t = \mathcal{X}_t^*(p_t)$. The Lagrangian associated with this problem is given by

$$L = \sum_t \beta^t \Big[u(c_t, m_t) + \phi_t(y_t - c_t - p_t m_t) + \mu_t(\mathcal{X}_t^*(p_t) - m_t) \Big],$$

with first-order conditions $u_{c,t} = \phi_t$ for domestic consumption, $u_{m,t} = p_t\phi_t + \mu_t$ for imports, and $p_t\phi_t = \mu_t \frac{\partial \mathcal{X}_t^*}{\partial p_t}$ for prices. Solving out for the multiplier μ_t we can write

$$u_{m,t} = p_t\phi_t + \frac{1}{\frac{\partial \mathcal{X}_t^*}{\partial p_t}} m_t \, \phi_t,$$

where the Lagrange multiplier $\phi_t = u_{c,t}$ represents the utility value of one additional unit of numeraire (domestic consumption), which is marginal utility $u_{c,t}$. This equation tells us that the Home planner trades off the utility benefit of consuming imports against a direct monetary cost and an indirect cost due to equilibrium price adjustment. A unit of imports delivers $u_{m,t}$ utility units. It costs p_t units of domestic consumption, which would alternatively deliver a total of $p_t u_{c,t}$ utility units—the direct cost. The Home planner also internalizes, however, that increasing imports by one unit is only implementable by allowing the price p_t to rise by exactly $1/\frac{\partial \mathcal{X}_t^*}{\partial p_t}$ to incentivize Foreign to supply one additional unit of exports. This increase in price raises the cost of inframarginal imports m_t. And the alternative use of funds is again domestic consumption, delivering utility units in proportion to $u_{c,t}$. That is, the RHS captures the total cost in utility units of consuming one additional unit of imports.

Putting all this together, we arrive at Home's social optimality condition for imports

$$u_{m,t} = p_t u_{c,t}\Big(1 + \frac{1}{\varepsilon_t}\Big) \tag{4}$$

written in terms of the static export supply elasticity $\varepsilon_t = \frac{\partial \log \mathcal{X}_t^*}{\partial \log p_t} = \frac{m_t}{p_t}\frac{\partial \mathcal{X}_t^*}{\partial p_t}$. When households in Home make consumption decisions, they do not internalize the indirect cost of imports through equilibrium price impact. They choose import demand to satisfy the private first-order condition $u_{m,t} = p_t u_{c,t}$. However, the Home planner can make households internalize the price impact by confronting them with an import tariff, so that the effective price of imports households face is instead $p_t(1+\tau_t)$. By setting τ_t to exactly match the social optimality condition (4), we arrive at the classic tariff formula (1),

$$\tau_t = \frac{1}{\varepsilon_t}.$$

The validity of the classic tariff formula therefore relies on two key assumptions: the static budget constraint of Home households and the existence of a static export supply function representing Foreign. Together, these two assumptions shut down important mechanisms of intertemporal linkages that are found in modern, micro-founded models of intertemporal trade.

2.2 Environment

We now introduce a large class of dynamic heterogeneous agent (HA) environments and derive our intertemporal tariff formula. There are two large countries, Home and Foreign. Home is populated by a measure one continuum of households, indexed by $i \in \mathcal{I} = [0,1]$. Time is discrete and indexed by $t \in \{0, 1, \ldots\}$. At the beginning of each period t, a stochastic event s_t realizes. We denote the history of such events by $s^t = (s_0, s_1, \ldots, s_t)$ with probability $\pi(s^t)$. We allow for idiosyncratic uncertainty but abstract from aggregate uncertainty. There is a single consumption good at each date and history. We study the case with multiple tradable goods and both intra- and intertemporal trade in Section 4.

Preferences. The lifetime utility of individual i in Home is defined as

$$V_0^i = \sum_t \beta^t \sum_{s^t} \pi(s^t) u^i(c_t^i(s^t), \ell_t^i(s^t)), \tag{5}$$

where $c_t^i(s^t)$ and $\ell_t^i(s^t)$ denote i's consumption and labor supply in history s^t.

Technologies. Home is endowed with a production technology to produce the single consumption good at date t using labor,

$$y_t = F_t(\ell_t), \tag{6}$$

where y_t denotes aggregate output and ℓ_t total use of effective labor in production. The production function $F_t(\cdot)$ is allowed to depend on time, which may for example reflect deterministic time variation in technology. In the main text, we study environments with a single factor for simplicity. In the Appendix, we show that our approach applies to richer models of production featuring intertemporal linkages (capital) and intermediate inputs.

Resource constraints. The resource constraint for effective labor in Home is given by

$$\ell_t = \int_0^1 z_t^i(s^t)\ell_t^i(s^t)di. \tag{7}$$

Household i's hours of work $\ell_t^i(s^t)$ are transformed into effective labor by i's productivity shifter $z_t^i(s^t)$ at date t and history s^t. We assume that $z_t^i(s^t)$ follows an exogenous Markov chain for each individual i and is therefore a source of idiosyncratic uncertainty. Since we abstract from aggregate uncertainty, a law of large numbers holds and aggregate effective labor ℓ_t is not contingent on the realization of history s^t at date t.

Finally, the resource constraint for the consumption good at date t is given by

$$x_t^* = \int_0^1 c_t^i(s^t)di - y_t, \tag{8}$$

where x_t^* is Foreign's aggregate export supply. We denote Foreign variables with asterisks. Foreign exports must equal Home's aggregate imports, that is, aggregate consumption $c_t = \int_0^1 c_t^i(s^t)di$ net of output y_t.

Intertemporal export supply function. We denote the intertemporal price of consumption at date t by p_t. Anticipating our use of sequence-space methods (Auclert et al., 2021), we use bold-faced notation for the infinite sequence of prices $\boldsymbol{p} = \{p_t\}_{t\geq 0}$. Our key assumption in this section is that there exists an *intertemporal export supply function* $\mathcal{X}^* : \ell^\infty \to \ell^\infty$ that is (Fréchet) differentiable and satisfies

$$x_t^* = \mathcal{X}_t^*(\boldsymbol{p}) \qquad \text{and} \qquad NFA_0^* + \sum_t p_t \mathcal{X}_t^*(\boldsymbol{p}) = 0. \tag{9}$$

The intertemporal export supply function $\mathcal{X}^*$ is a sequence-space function analogous to the intertemporal consumption functions used widely in macroeconomics. It maps an infinite sequence of prices $\boldsymbol{p}$ to a level of Foreign export supply at each date t. The micro-foundations of $\mathcal{X}^*$ we study in this paper all assume budget constraints and no-Ponzi conditions at the micro level that, when aggregated, imply the external balance condition in (9), with NFA_0^* denoting Foreign's initial aggregate net foreign asset position.

Our strategy in this section is to work directly with $\mathcal{X}^*$ as a reduced-form representation of Foreign, not taking a stance on its underlying micro-foundations. We show that *intertemporal export supply elasticities*—the sequence-space Jacobian of $\mathcal{X}^*$—are a sufficient statistic for Home's intertemporal terms of trade manipulation motive and thus for the optimal tariff that decentralizes Home's efficient allocation. Our main result in this section shows that as long as there exists a function $\mathcal{X}^*$ that satisfies (9), the efficient tariff takes a very particular shape. In Section 3, we will then study different micro-foundations for $\mathcal{X}^*$ and show that its sequence-space Jacobian differs qualitatively across several important benchmark models of intertemporal trade.

2.3 Home Efficiency

We now characterize efficient allocations from the perspective of Home. Our notion of efficiency in this paper is that of Pareto efficiency, so that a Home allocation is efficient if there are no feasible perturbations that make one individual *at Home* better off without leaving others at Home worse off. Such allocations must be feasible—they must satisfy technologies and resource constraints—but additionally they must be implementable—accounting for the fact that Home can exploit its market power in trade.

We take a dual approach in this paper and keep the sequence of prices $\boldsymbol{p}$ explicit in our formulation of the planning problem below. This contrasts with the primal approach often used in the literature (Costinot et al., 2014). The primal approach would first solve for the price as a function of the Home allocation and then work with an implementability condition specified in terms of the Home allocation. We find the dual approach useful for two reasons. First, the efficiency condition and optimal tariff formula it delivers generalize directly to a broad class of environments. And second, it allows us to characterize optimal tariffs in terms of export supply elasticities in the tradition of prior work in the trade literature.

Lemma 1 (Implementability). *A Home allocation $\{c_t^i(s^t), \ell_t^i(s^t)\}_{i,t,s^t}$ and prices $\boldsymbol{p}$ are implementable if and only if they satisfy*

$$\mathcal{X}_t^*(\boldsymbol{p}) = \int_0^1 c_t^i(s^t)di - F_t\left(\int_0^1 z_t^i(s^t)\ell_t^i(s^t)di\right) \tag{10}$$

for all t.

Efficient allocations therefore solve the planning problem

$$\max \int_0^1 \alpha^i V_0^i \, di \quad \text{s.t.} \quad (10)\,. \tag{11}$$

Varying the Pareto weights α^i allows us to trace out the Pareto frontier. In other words, we say that an allocation and prices are efficient from the perspective of Home if there exist weights $\{\alpha^i\}_i$ such that they solve Home's efficiency problem (11). The Lagrangian associated with this planning problem is given by

$$L = \int_0^1 \alpha^i \sum_t \beta^t \sum_{s^t} \pi(s^t) u^i(c_t^i(s^t), \ell_t^i(s^t))\, di + \sum_t \beta^t \mu_t \left[\mathcal{X}_t^*(\boldsymbol{p}) - \int_0^1 c_t^i(s^t)di + F_t\left(\int_0^1 z_t^i(s^t)\ell_t^i(s^t)di\right)\right],$$

where we denote by μ_t the Lagrange multiplier on the date t implementability condition. In the classic efficiency problem of a closed economy, the planner maximizes social welfare subject to technologies and resource constraints. If the Home planner took Foreign exports $\boldsymbol{x}^*$ as given, the Home efficiency problem (11) and the associated Lagrangian L would collapse to the classic efficiency problem. Here, the Home planner also internalizes the price impact that choosing one Home allocation over another has in the world market for tradable goods, as captured by the implementability condition (10). We present the complete set of first-order conditions for problem (11) and derive the Home efficiency conditions in Appendix A.2. The following Proposition summarizes and presents the efficiency condition for Home's intertemporal terms of trade.

Proposition 1 (Home Efficiency)**.** *A Home allocation $\{c_t^i(s^t), \ell_t^i(s^t)\}_{i,t,s^t}$ and prices $\boldsymbol{p}$ are efficient if they satisfy (i) the classic efficiency conditions (Mas-Colell et al., 1995) and (ii) the intertemporal terms of trade efficiency condition*

$$0 = \sum_k MRS_{kt} \frac{\partial \mathcal{X}_k^*}{\partial p_t}, \tag{12}$$

where MRS_{kt} is Home's marginal rate of substitution between consumption at dates k and t.

Proposition 1 characterizes the necessary conditions for a Pareto efficient allocation and prices from the perspective of Home. It generalizes the classic efficiency conditions to a two-country environment where Home internalizes its price impact in the world market for tradable goods.

The Home efficiency problem (11) allows the planner to pick the Home allocation freely, subject only to technologies and resource constraints. So the Home planner will ensure that the allocation satisfies the usual efficiency conditions *internally*—requiring that all marginal rates of substitution (MRS) are equalized, and that MRS are equal to marginal rates of transformation (MRT) for the allocation internal to Home. Appendix A.2 presents a self-contained characterization of these conditions for completeness.

If Home was a closed economy and $\boldsymbol{x}^*$ represented an exogenous endowment sequence taken as given by the Home planner, then the classic efficiency conditions would fully characterize Pareto efficient allocations. However, when Home internalizes that choosing one domestic allocation over another affects the world prices of tradable goods, equation (12) emerges as a new efficiency condition for Home's intertemporal terms of trade.

The Home planner cannot choose $\boldsymbol{x}^*$ freely but from a possibility frontier in the space of sequences. In the dual representation, this possibility frontier is defined as the set of aggregate imports sequences that are implementable according to (10) by a valid sequence of prices, that is, the set of sequences

$$\left\{ \boldsymbol{c} - \boldsymbol{y} \in \ell^\infty : \text{ there exists } \boldsymbol{p} \gg 0 \text{ with } c_t - y_t = \mathcal{X}_t^*(\boldsymbol{p}) \right\}$$

Intuitively, the Home planner understands that choosing a particular allocation of consumption

$\{c_t^i(s^t)\}$ and labor supply $\{\ell_t^i(s^t)\}$ implies a sequence of aggregate import demand, $c_t - y_t$. This import sequence must be implementable, in the sense that there must be a sequence $\boldsymbol{p}$ that solves (10)—that is, a sequence of prices at which Foreign is willing to supply the necessary exports to clear the world market for tradable goods. The Home planner understands that she can move along this frontier and choose her desired point on the frontier by choosing the associated price sequence. This optimal point on the frontier is precisely characterized by the new efficiency condition (12). And once the planner has chosen where on this frontier to locate, then the rest of the allocation is pinned down by the usual efficiency conditions taking as given the associated sequence of Foreign exports.

In the dual representation of this problem, the Home planner directly chooses a price sequence $\boldsymbol{p}$ and understands that this choice of prices pins down Foreign's desired export supply $\boldsymbol{x}^* = \mathcal{X}^*(\boldsymbol{p})$. The intuition behind equation (12) in the dual representation is surprisingly similar to the intuition behind the classic tariff formula we discussed in Section 2.1. Consider the perturbation of marginally increasing the price of consumption p_t at date t. Since Foreign's export supply function $\mathcal{X}^*$ is smooth by assumption, the increase in price p_t induces Foreign to change its export supply in all periods k by $\frac{\partial \mathcal{X}_k^*}{\partial p_t}$. Implementability (10) then requires that Home increases its aggregate import demand $c_k - y_k$ by $\frac{\partial \mathcal{X}_k^*}{\partial p_t}$ in all periods k. Starting from an efficient allocation and prices, it must be that the Home planner cannot increase social welfare by marginally increasing or decreasing price p_t: the marginal impact of such a perturbation on the Lagrangian L must be 0. Therefore, the marginal impact on social welfare of changing aggregate Home imports by $\frac{\partial \mathcal{X}_k^*}{\partial p_t}$ in all periods k must be 0.

What, then, is the welfare impact of a marginal increase in Home imports at date k? Intuitively, the planner can allocate an increase (decrease) in imports to the individual with the highest (lowest) marginal (social) utility of consumption, $\alpha^i \beta^k \pi(s^k) u_{c,k}^i(s^k)$. Whenever the classic efficiency conditions are satisfied, however, marginal social utility of consumption is equalized across all individuals i. This measure of social utility still features the Pareto weight α^i, which we only use to trace out the Pareto frontier. To get rid of it, we divide by social marginal utility at date t, yielding an MRS between consumption at dates k and t, exactly as it appears in equation (12). Finally, we sum over all periods k in which the price perturbation ∂p_t affects Foreign exports, arriving at equation (12).

2.4 Intertemporal Tariff Formula

We now ask how the efficient allocations of Proposition 1 might be decentralized. However, we do not take a stance on the set of instruments required to satisfy the classic efficiency conditions—these will depend on the exact details of the environment. Instead, we show that an import tariff can be set to satisfy the efficiency condition (12) for Home's intertemporal terms of trade whenever the classic efficiency conditions are satisfied. Our first main result is a formula that characterizes these tariffs in terms of intertemporal export supply elasticities. We now derive this formula constructively in three steps.[2]

First, an import tariff τ_t represents an ad-valorem consumption tax combined with an ad-valorem production subsidy at date t. At any allocation and prices that satisfy the classic efficiency conditions, households in Home therefore face the intertemporal consumption prices $(1+\tau_t)p_t$, and we must have

$$MRS_{kt} = \frac{(1+\tau_k)p_k}{(1+\tau_t)p_t}.$$

In other words, the MRS between consumption at dates k and t is equalized across all households, and it must be equal to the relative price these households face. In order for the Home efficiency condition (12) to be satisfied, we therefore must have

$$0 = \sum_k \frac{(1+\tau_k)p_k}{(1+\tau_t)p_t} \frac{\partial \mathcal{X}_k^*}{\partial p_t} = \sum_k (1+\tau_k)p_k \frac{\partial \mathcal{X}_k^*}{\partial p_t} = \sum_k p_k \frac{\partial \mathcal{X}_k^*}{\partial p_t} + \sum_k \tau_k p_k \frac{\partial \mathcal{X}_k^*}{\partial p_t}.$$

Second, the intertemporal export supply function $\mathcal{X}^*$ satisfies Foreign's external balance condition (9) by assumption. Given an initial NFA position NFA_0^*, this condition holds for any strictly positive price sequence $\boldsymbol{p}$. Therefore, we can differentiate this equation with respect to p_t, which implies

$$0 = x_t^* + \sum_k p_k \frac{\partial \mathcal{X}_k^*}{\partial p_t}.$$

Finally, we put these two equations together and rearrange,

$$x_t^* = \sum_k \tau_k p_k \frac{\partial \mathcal{X}_k^*}{\partial p_t} = \sum_k \tau_k p_k \frac{x_k^*}{p_t} \frac{\partial \log \mathcal{X}_k^*}{\partial \log p_t}.$$

Multiplying and dividing by τ_t now yields our result, which we summarize in the following Proposition.

Proposition 2 (Intertemporal Tariff Formula)**.** *As part of a decentralization of the Home efficiency allocation, the efficiency condition (12) for intertemporal terms of trade is satisfied if Home consumers face an ad-valorem import tariff given by*

$$\frac{1}{\tau_t} = \sum_k \omega_{kt} \frac{\partial \log \mathcal{X}_k^*}{\partial \log p_t} \tag{13}$$

where ω_{kt} represents Home's relative tariff revenue weight

$$\omega_{kt} = \frac{\tau_k p_k x_k^*}{\tau_t p_t x_t^*}.$$

[2] Our approach in this subsection admits two useful interpretations. The tariff formula we derive is valid whenever the classic efficiency conditions are satisfied at Home. This may be the case because the Home economy is internally efficient to begin with, as will be the case in the application we consider in Section 3.1, for example. Alternatively, the Home planner may have access to a sufficiently large set of instruments that allows her to implement the efficient allocation. We will showcase this in the New Keynesian application we consider in Appendix C, where monetary policy is sufficient to ensure production efficiency.

Proposition 2 presents an intertemporal tariff formula that characterizes the optimal tariff in terms of two sufficient statistics: intertemporal export supply elasticities $\frac{\partial \log \mathcal{X}_k^*}{\partial \log p_t}$ and relative tariff revenue weights ω_{kt}. Intertemporal export supply elasticities are captured by the sequence-space Jacobian of $\mathcal{X}^*$ (Auclert et al., 2021), the infinite matrix $\boldsymbol{\mathcal{E}}^*$ with entries

$$\boldsymbol{\mathcal{E}}_{kt}^* = \frac{\partial \log \mathcal{X}_k^*}{\partial \log p_t}$$

in row k and column t. Denoting by $\boldsymbol{\mathcal{E}}_t^*$ the t-th column of this matrix, the intertemporal tariff formula can also be written as

$$\tau_t = \frac{1}{\boldsymbol{\omega}_t' \boldsymbol{\mathcal{E}}_t^*}, \tag{14}$$

where $\boldsymbol{\omega}_t'$ is the t-row of the tariff revenue weight matrix $\boldsymbol{\omega}$. In the tradition of recent sequence-space results in the heterogeneous agent macro literature, we identify intertemporal export supply elasticities—or *iXEs*—as key determinants of optimal tariffs, similar to Auclert et al. (2024b)'s generalization of the static Keynesian cross to an intertemporal one using intertemporal marginal propensities to consume (iMPCs).

According to the classic tariff formula (1), the contemporaneous export supply elasticity ε_t is a sufficient statistic for the optimal tariff. This would be a valid characterization if the matrix of iXEs $\boldsymbol{\mathcal{E}}^*$ was diagonal. Proposition 2 generalizes the classic tariff formula to dynamic environments that admit a representation of Foreign in terms of the intertemporal export supply function $\mathcal{X}^*(\boldsymbol{p})$. In such settings, both the diagonal and off-diagonal entries of the elasticity matrix $\boldsymbol{\mathcal{E}}^*$ determine the optimal tariff. It is therefore clear that the structure of this matrix is key to understanding Home's intertemporal terms of trade manipulation motive. Section 3 presents three alternative micro-foundations of $\boldsymbol{\mathcal{E}}^*$ and investigates its structure across important benchmark models of intertemporal trade.

2.5 Second-Best Tariffs

The intertemporal tariff formula we presented in Proposition 2 characterizes the optimal tariff in the absence of distortions in the Home economy. In this case, tariffs are used for a singular purpose: intertemporal terms of trade manipulation. And because marginal rates of substitution are equalized across households, the planner values changes in net exports across periods at the economy-wide MRS_{kt}. When the first-best allocation is not achievable, however, this logic fails for two reasons. First, MRS are no longer equalized across individuals, which complicates determining the planner's valuation of changes in imports x_t^* at a given date. Second, it is now welfare-improving to use tariffs as a second-best instrument to target other inefficiencies in the economy.

In this subsection, we develop a general theory of second-best optimal tariffs in dynamic heterogeneous agent economies. Our main result is a targeting rule that identifies the normative considerations relevant for second-best tariffs.

Constrained planning problem. In Sections 2.3 and 2.4, we considered the Home efficiency problem, under which the planner can directly choose any feasible Home allocation $\{c_t^i(s^t), \ell_t^i(s^t)\}$ and price sequence $\boldsymbol{p}$ that satisfy the implementability condition (10). In this section, we instead consider a Ramsey problem, under which the planner has access to a sequence of policy instruments $\boldsymbol{\tau} = \{\tau_t\}_{t\geq 0}$ but faces a set of constraints rich enough to nest a large class of heterogeneous agent models. We continue to assume the existence of a Foreign export supply function $x_t^* = \mathcal{X}_t^*(\boldsymbol{p})$ that maps a sequence of prices $\boldsymbol{p}$ to the level of exports at date t and satisfies equation (9).

In particular, we assume that there exist consumption and labor supply functions that determine individual i's consumption and labor supply at date t and history s^t according to

$$c_t^i(s^t) = \mathcal{C}_t^i(\boldsymbol{p}, \boldsymbol{\tau}, s^t) \qquad \text{and} \qquad \ell_t^i(s^t) = \mathcal{L}_t^i(\boldsymbol{p}, \boldsymbol{\tau}, s^t). \tag{15}$$

Similar to Foreign's export supply function, $\mathcal{C}_t^i(\cdot)$ and $\mathcal{L}_t^i(\cdot)$ are sequence-space functions, in the sense that they map infinite sequences of prices $\boldsymbol{p}$ and policy instruments $\boldsymbol{\tau}$ to levels of individual consumption and labor supply. Such functions are commonly used in macroeconomics to characterize heterogeneous agent models (Auclert et al., 2024b). Crucially, equations (15) imply that the planner can no longer choose consumption and labor supply freely, subject only to technologies and resource constraints, but must also respect the constraints embedded in $\mathcal{C}_t^i(\cdot)$ and $\mathcal{L}_t^i(\cdot)$.

A consumption function of the form (15) emerges naturally in standard incomplete markets models of intertemporal consumption-smoothing. In particular, the functions $\mathcal{C}_t^i(\cdot)$ and $\mathcal{L}_t^i(\cdot)$ may encode not only household preferences, budget constraints and borrowing limits, but also competitive general equilibrium conditions that are internal to the Home economy.

Instead of choosing consumption and labor supply directly as before, the planner can now only affect them indirectly through the policy instruments $\boldsymbol{\tau}$. The Ramsey problem is therefore to maximize social welfare subject to technologies, resource constraints, the external implementability condition (10), as well as the new internal implementability conditions (15). It is given by

$$\max \int_0^1 \alpha^i \sum_t \beta^t \sum_{s^t} \pi(s^t) u^i(c_t^i(s^t), \ell_t^i(s^t))\, di,$$

subject to

$$c_t^i(s^t) = \mathcal{C}_t^i(\boldsymbol{p}, \boldsymbol{\tau}, s^t)$$

$$\ell_t^i(s^t) = \mathcal{L}_t^i(\boldsymbol{p}, \boldsymbol{\tau}, s^t)$$

$$\mathcal{X}_t^*(\boldsymbol{p}) = \int_0^1 c_t^i(s^t)\, di - F_t\left(\int_0^1 z_t^i(s^t)\ell_t^i(s^t)\, di\right)$$

Notice that we assume Foreign's export supply remains a function of the world prices $\boldsymbol{p}$ only. This is consistent with the interpretation of τ_t as an import tariff on Home consumers at date t, which does not affect Foreign directly.

Wedges. It will be useful to introduce notation for the relevant wedges between the Home efficiency and second-best allocations. In the main text, we focus on environments with no ex-ante or permanent heterogeneity. In other words, all individuals i are homogeneous at date 0 but become different over time (ex post heterogeneity) due to the realization of idiosyncratic shocks $z_t^i(s^t)$. In particular, this assumption rules out that second-best optimal policy is motivated by an explicit redistribution motive. We consider the more general case of ex ante and ex post heterogeneity in the Appendix.

In the absence of ex ante heterogeneity, there are three relevant wedges that can emerge in second-best allocations. First, we denote deviations from the Home efficiency condition for intertemporal terms of trade by

$$\Lambda_t^{\mathrm{ITM}} = \sum_k \omega_k \frac{\partial \mathcal{X}_k^*}{\partial p_t}, \tag{16}$$

where we denote by

$$\omega_k = \int_0^1 \frac{\beta^k \sum_{s^k} \pi(s^k) u_{c,k}^i(s^k)}{u_{c,0}^i} di \tag{17}$$

the relevant social valuation of an increase in Home imports at date k. In particular, $\beta^k \sum_{s^k} \pi(s^k) u_{c,k}^i(s^k)$ denotes i's valuation of a unit increase in consumption in all histories s^k at date k. The term under the integral sign therefore corresponds to i's marginal rate of substitution between a unit of consumption at date k (in all histories) and consumption at date 0. We obtain ω_k by averaging this MRS across individuals, which turns out to be the appropriate notion of social MRS between consumption at dates k and 0.

Second, we define the aggregate labor wedge in the spirit of Farhi and Werning (2016) as

$$\Lambda_t^\ell = 1 + \frac{u_{\ell,t}^i(s^t)}{u_{c,t}^i(s^t)} \frac{1}{z_t^i(s^t) F_{\ell,t}}, \tag{18}$$

where $F_{\ell,t} = F_t'(\ell_t)$ is the derivative of the production function with respect to labor at date t. Conditional on the aggregate labor wedge, we assume for simplicity that households are then on their individual labor-leisure conditions. In other words, we assume that $\mathcal{L}_t^i(\cdot)$ is such that the RHS of equation (18) is equalized across all individuals at all dates t and histories s^t.[3]

[3] Intuitively, if there is a common wage w_t that all households face, as will be the case in our applications, then individual labor-leisure conditions imply $-u_{\ell,t}^i(s^t) = w_t z_t^i(s^t) u_{c,t}^i(s^t)$ and the aggregate labor wedge simply becomes $\Lambda_t^\ell = 1 - \frac{w_t}{A_t}$.

Third and finally, we introduce wedges in individual risk-sharing, defined as

$$\Lambda_t^i(s^t) = \frac{\pi(s^t)u'(c_t^i(s^t))}{\sum_{s^t} \pi(s^t)u'(c_t^i(s^t))} - 1. \tag{19}$$

Intuitively, if financial markets were complete and individuals could fully insure themselves, they would smooth private marginal utility across histories s^t at a given date t. This condition is satisfied at an efficient allocation and implies $\pi(s^t)u'(c_t^i(s^t)) = \pi(\tilde{s}^t)u'(c_t^i(\tilde{s}^t))$ for any two histories s_t and $\tilde{s}^t$.

In summary, Λ_t^{ITM}, Λ_t^ℓ and $\Lambda_t^i(s^t)$ represent wedges in the sense that they are 0 at efficient allocations. With these wedges in hand, we can now present a targeting rule for optimal second-best tariffs.

Proposition 3 (Second-Best Tariffs)**.** *In the absence of ex ante heterogeneity, the optimal second-best tariff trades off intertemporal terms of trade manipulation against production efficiency and risk-sharing. The optimality condition for tariff* τ_k *is*

$$0 = \underbrace{\sum_t \Lambda_t^{ITM} \frac{dp_t}{d\tau_k}}_{\substack{\text{Intertemporal} \\ \text{Terms of Trade Manipulation}}} + \underbrace{\sum_t \omega_t \Lambda_t^\ell \frac{dy_t}{d\tau_k}}_{\text{Production Efficiency}} + \underbrace{\sum_t \omega_t \sum_{s^t} \mathbb{C}ov_i \left(\Lambda_t^i(s^t), \frac{dV_t^i(s^t)}{d\tau_k} \right)}_{\text{Risk-Sharing}}, \tag{20}$$

where we denote by $\frac{dV_t^i(s^t)}{d\tau_k} = \frac{dc_t^i(s^t)}{d\tau_k} - z_t^i(s^t)F_{\ell,t}(1 - \Lambda_t^\ell)\frac{d\ell_t^i(s^t)}{d\tau_k}$ *the consumption-equivalent welfare gain of individual* i *at date* t *history* s^t *from the tariff perturbation* τ_k*. The terms of trade wedge* Λ_t^{ITM}*, the labor wedge* Λ_t^ℓ*, and the risk-sharing wedge* $\Lambda_t^i(s^t)$ *are defined in* (16), (18), *and* (19).

Proposition 3 characterizes a targeting rule for second-best tariffs in dynamic heterogeneous agent economies. When the first-best allocation is not achievable, the efficient tariff formula of Proposition 2 fails for two reasons: First, dispersion in individuals' marginal utilities at date t affects the planner's valuation of a change in date t net exports. In this case, ω_k is the relevant generalization of the social MRS between dates k and 0. And second, the planner finds it valuable to use tariffs not only for intertemporal terms of trade manipulation but also to tackle other inefficiencies. Equation (20) illustrates how the optimal second-best tariff is shaped by both of these forces.

At a second-best allocation, the optimal tariff trades off two new motives against intertemporal terms of trade manipulation: production efficiency and risk-sharing. We discuss each of the three terms in equation (20) in turn. The first term, $\sum_t \Lambda_t^{\text{ITM}} \frac{dp_t}{d\tau_k}$ captures the same terms of trade manipulation forces already characterized in equation (12). At an efficient allocation, condition (12) tells us that the planner chooses p_t directly to set $\Lambda_t^{\text{ITM}} = 0$. Here, the planner can no longer choose world prices p_t directly subject only to the external implementability condition (10).

Instead, the planner must also respect internal implementability (15) and can influence world prices only indirectly by choosing tariffs τ_k. In other words, the terms of trade gain from a marginal increase in tariff τ_k at date t is determined as follows. First, the change in tariff $d\tau_k$ directly affects Home households desired consumption and labor supply through $\mathcal{C}_t^i(\cdot)$ and $\mathcal{L}_t^i(\cdot)$. These changes aggregate into a change in Home's aggregate import demand, which must be accommodated by Foreign export supply to satisfy the implementability condition (10). In general equilibrium, this will induce price changes $\frac{dp_t}{d\tau_k}$. These price changes will, in turn, generate gains from terms of trade in proportion to Λ_t^{ITM}.

The second term describes the production efficiency gains from a change in tariff $d\tau_k$. The change in tariff directly and indirectly (through GE changes in world prices $\boldsymbol{p}$) affects households' labor supply through $\mathcal{L}_t^i(\cdot)$. This, in turn, leads to a change in Home's aggregate output, $\frac{dy_t}{d\tau_k}$. Labor wedges Λ_t^ℓ represent a measure of how depressed production in a given period t is (Farhi and Werning, 2016). So if tariffs stimulate production in periods where activity is inefficiently low, then this results in production efficiency welfare gains. Finally, these gains are discounted at the social MRS ω_t.

And finally, tariffs now potentially generate efficiency gains from improved risk-sharing. A change in tariff $d\tau_k$ directly and indirectly (through the equilibrium response of world prices $\boldsymbol{p}$) affects individual i's consumption at all dates t and histories s^t. To the extent that tariffs raise the consumption of those individuals whose weights $\Lambda_t^i(s^t)$ are relatively high, tariffs help provide ex post consumption smoothing across individuals and therefore improves risk-sharing. These gains are again discounted at the relevant social MRS ω_t.

We conclude with one final but important observation: Second-best tariffs now trade off several sources of welfare gains and losses. However, the intertemporal export supply function $\mathcal{X}^*$ and its sequence-space Jacobian are still key determinants of the terms of trade manipulation motive of second-best tariffs through the term Λ_t^{ITM}. The only change relative to Sections 2.3 and 2.4 is that the planner must now also take into consideration how tariffs at date k affect equilibrium prices p_t at other dates t.[4]

3 Models of Intertemporal Export Supply Elasticities

Intertemporal export supply elasticities are sufficient statistics for optimal tariffs in dynamic environments. We now present three alternative micro-foundations of the intertemporal export supply function $\mathcal{X}^*$ and its sequence-space Jacobian. We show in Section 3.1 that intertempral export supply elasticities take a special form in representative agent (RA) economies because households are infinitely forward-looking. In fact, the optimal tariff formula collapses to a static one in these environments. This is no longer the case in the presence of household heterogeneity

[4] Models with more than two countries may still admit a sequence-space representation $\mathcal{X}^*(\boldsymbol{p})$ for aggregate rest of the world export supply. In that case, our results in this section generalize directly to many-country environments.

and incomplete markets, as we show in Section 3.2. Finally, we present a New Keynesian model in Appendix C and show how intertemporal export supply elasticities are shaped by Foreign business cycle conditions. Our intertemporal tariff formula (13) and targeting rule (20) provide a unified framework that allows us to contrast optimal tariffs across these different environments.

3.1 A Representative Agent Endowment Economy

The first model of intertemporal export supply elasticities we consider is a deterministic representative agent (RA) endowment economy. Working through this simple benchmark is useful to set the stage and take a first look at $\mathcal{X}^*$ and its sequence-space Jacobian. It also allows us to revisit Costinot et al. (2014), who study the same environment, and clarify how our approach relates to theirs. Our main result in Section 3.1 is that the intertemporal export supply function $\mathcal{X}^*$ and its derivatives take a special form in RA environments with permanent-income consumers.[5]

3.1.1 Environment

We start by describing the world competitive equilibrium taking Home import tariffs as given, and then apply Proposition 2 directly.

Households. Home and Foreign are each populated by representative households whose lifetime utilities are

$$V_0 = \sum_t \beta^t u(c_t) \qquad \text{and} \qquad V_0^* = \sum_t \beta^t u(c_t^*), \tag{21}$$

where c_t and c_t^* denote each household's consumption of the single consumption good at date t. This good appears as an endowment in both countries, denoted y_t and y_t^*. As before, we use bold-faced notation for infinite sequences such as $\boldsymbol{y} = \{y_t\}_{t\geq 0}$.

Assuming complete financial markets allows us to write the consumption-savings problems of both households in terms of the lifetime budget constraints,

$$NFA_0 = \sum_t \Big[(1+\tau_t)p_t(c_t - y_t) - T_t\Big] \qquad \text{and} \qquad NFA_0^* = \sum_t p_t(c_t^* - y_t^*), \tag{22}$$

where p_t denotes the intertemporal price of date t consumption, τ_t is Home's import tariff (export subsidy), and T_t is a lump-sum rebate. We take as given both countries' initial net foreign asset (NFA) positions NFA_0 and NFA_0^*, and we assume that Foreign does not set a tariff.

Households maximize preferences (21) subject to (22). Denoting the Lagrange multipliers on each lifetime budget constraint by λ and λ^*, the associated first-order conditions are

$$\beta^t u'(c_t) = \lambda(1+\tau_t)p_t \qquad \text{and} \qquad \beta^t u'(c_t^*) = \lambda^* p_t. \tag{23}$$

[5] We present a RA model with production and an endogenous labor supply choice in Appendix B and show that our main insight about the sequence-space Jacobian of $\mathcal{X}^*$ remains unchanged in that environment.

Government, market clearing, and competitive equilibrium. The Home government runs a balanced budget. This requires that import tariff revenue (export subsidy outlays) equal lump-sum transfers (taxes),

$$\tau_t p_t (c_t - y_t) = T_t \tag{24}$$

in all periods t. In other words, $\tau_t > 0$ represents a tax on imports when $c_t - y_t > 0$ and a subsidy on exports when $c_t - y_t < 0$. The Foreign government is passive and does not set tariffs.

The market clearing condition for the single consumption good at date t is given by

$$y_t + y_t^* = c_t + c_t^*. \tag{25}$$

We now define a competitive equilibrium with tariffs.

Definition 1 (Competitive Equilibrium)**.** *Taking as given initial net foreign asset positions NFA_0 and NFA_0^*, endowment sequences $\boldsymbol{y}$ and $\boldsymbol{y}^*$, as well as Home tariffs $\boldsymbol{\tau}$, a competitive equilibrium comprises an allocation $(\boldsymbol{c}, \boldsymbol{c}^*, \boldsymbol{T})$, multipliers (λ, λ^*) and prices $\boldsymbol{p}$ that satisfy lifetime budget constraints* (22)*, household first-order conditions* (23)*, the Home government budget constraint* (24)*, and market clearing* (25)*.*

Notice that if Home took consumption prices p_t as given, the Home allocation without policy would be efficient. This allows us to apply the results from Sections 2.3 and 2.4 directly.

3.1.2 Intertemporal Export Supply Elasticities

The Home efficiency condition (12) and optimal tariff formula (13) apply as-is to this environment. The key question is therefore how to characterize Foreign's intertemporal export supply function $\mathcal{X}^*$. We first present a constructive derivation of $\mathcal{X}^*$ and then characterize its sequence-space Jacobian.

Foreign export supply $x_t^* = y_t^* - c_t^*$ is fully determined by the consumption behavior of the Foreign representative consumer since the endowment y_t^* is exogenous. And the Foreign household problem can be characterized fully by the lifetime budget constraint (22) and the first-order condition in (23). Taking as given a price sequence $\boldsymbol{p}$, these two equations solve for the Lagrange multiplier λ^* and a sequence of consumption $\boldsymbol{c}^*$. In particular, the date t first-order condition solves for c_t^* as a function of the contemporaneous price p_t and the Lagrange multiplier λ^*. And plugging back into the lifetime budget constraint solves for the multiplier λ^* as a function of the entire sequence of prices $\boldsymbol{p}$.

In summary, we can use the Foreign competitive equilibrium conditions to derive an intertemporal consumption function for Foreign, whose sequence-space representation is $c_t^* = \mathcal{C}_t^*(\boldsymbol{p})$. This consumption function $\mathcal{C}^*$ maps a time path of prices $\boldsymbol{p}$ into a time path of Foreign consumption $\boldsymbol{c}^*$ that is consistent with the optimality condition and lifetime budget constraint of the Foreign household. With Foreign's consumption function in hand, we can define the intertemporal export supply function simply as

$$\mathcal{X}_t^*(\boldsymbol{p}) = y_t^* - \mathcal{C}_t^*(\boldsymbol{p}).$$

We now show that the sequence-space Jacobian of $\mathcal{X}^*$ takes a special form because Foreign households are permanent-income consumers. First, notice that we have $\frac{\partial \mathcal{X}_k^*}{\partial p_t} = -\frac{\partial \mathcal{C}_k^*}{\partial p_t}$, which relates the Jacobian of the export supply function to that of the consumption function. In RA models with complete financial markets, the optimal consumption behavior of the Foreign household is fully determined by the lifetime budget constraint $0 = \sum_t p_t(c_t^* - y_t^*)$ and the first-order condition $\beta^t u'(c_t^*) = \lambda^* p_t$, where λ^* is the Lagrange multiplier on the lifetime budget constraint. We constructed the consumption function $\mathcal{C}_t^*(\boldsymbol{p})$ by plugging the FOC back into the lifetime budget constraint to solve for λ^* as a function of $\boldsymbol{p}$. Using the first-order condition directly, we can alternatively represent the consumption function as

$$c_t^* = \tilde{\mathcal{C}}_t^*(\lambda^*, p_t) \qquad \text{where} \qquad \mathcal{C}_t^*(\boldsymbol{p}) = \tilde{\mathcal{C}}_t^*(\lambda^*(\boldsymbol{p}), p_t).$$

In other words, Foreign consumption at date t depends directly only on the contemporaneous price p_t; it depends indirectly on all other prices through the Lagrange multiplier λ^*. This structure is special to the RA environment because we can represent the household problem in terms of a lifetime budget constraint. Therefore, we have

$$\frac{\partial \mathcal{C}_k^*}{\partial p_t} = \frac{\partial \tilde{\mathcal{C}}_k^*}{\partial \lambda^*}\frac{\partial \lambda^*}{\partial p_t} + \frac{\partial \tilde{\mathcal{C}}_k^*}{\partial p_t}\mathbb{1}_{k=t} = \frac{p_k}{\beta^k u''(c_k^*)}\frac{\partial \lambda^*}{\partial p_t} + \frac{\lambda^*}{\beta^k u''(c_k^*)}\mathbb{1}_{k=t}\,. \tag{26}$$

Foreign consumption at date k responds to a price change at date t for two reasons. First, there is a direct effect if $k = t$. An increase in the contemporaneous price induces consumers to spend less at date t—notice that $\frac{\lambda^*}{\beta^k u''(c_k^*)} < 0$ for all k. Second, there is an indirect effect. Since financial markets are complete, Foreign households consume the annuity value of their lifetime wealth in each period. The Lagrange multiplier λ^* on the lifetime budget constraint represents the value of a unit of lifetime wealth at date 0. A change in the price p_t of consumption at date t affects lifetime wealth and consequently consumption in all periods k. This indirect effect is captured by the first term in equation (26), and we have

$$\frac{\partial \lambda^*}{\partial p_t} = -\frac{\lambda^*}{\sum_s p_s}\left[\frac{u'(c_t^*)}{u''(c_t^*)} - x_t^*\right], \tag{27}$$

which follows from differentiating the lifetime budget constraint as we show in Appendix A.5. Finally, we denote by $\gamma_t^* = -c_t^* \frac{u''(c_t^*)}{u'(c_t^*)}$ Foreign's inverse elasticity of intertemporal substitution. This allows us to arrive at the following characterization, where we denote by $\boldsymbol{\phi}$ and $\boldsymbol{\rho}$ the infinite column vectors with t-th entries

$$\phi_t = -\frac{1}{\gamma_t^*} \qquad \text{and} \qquad \rho_t = \frac{\partial \log \lambda^*}{\partial \log p_t}.$$

Lemma 2. *Denote by $\boldsymbol{\mathcal{C}}$ the log sequence-space Jacobian of Foreign's intertemporal consumption function, i.e., the infinite matrix with entries $\boldsymbol{\mathcal{C}}_{kt} = \frac{\partial \log \mathcal{C}_k^*}{\partial \log p_t}$. It admits a decomposition into a diagonal matrix and a rank-one matrix,*

$$\boldsymbol{\mathcal{C}} = \boldsymbol{D} + \boldsymbol{P}\,, \qquad \textit{or equivalently} \qquad \frac{\partial \log \mathcal{C}_k^*}{\partial \log p_t} = \underbrace{-\frac{1}{\gamma_k^*}\mathbb{1}_{k=t}}_{\textit{Direct Effect}} \underbrace{- \frac{1}{\gamma_k^*}\frac{\partial \log \lambda^*}{\partial \log p_t}}_{\textit{Indirect Effect}},$$

where $\boldsymbol{D}$ is the diagonal matrix diag$(\boldsymbol{\phi})$ and $\boldsymbol{P} = \boldsymbol{\phi}\,\boldsymbol{\rho}'$ is the outer product of the two vectors $\boldsymbol{\phi}$ and $\boldsymbol{\rho}$. When consumption preferences $u(\cdot)$ are CRRA and $\gamma_t^* = \gamma$, then $\boldsymbol{\mathcal{C}}$ has two distinct eigenvalues, $-\frac{1}{\gamma}$ (with infinite multiplicity) and $-\frac{1}{\gamma}(1 + \sum_t \rho_t)$.

Lemma 2 establishes that the log sequence-space Jacobian $\boldsymbol{\mathcal{C}}$ has a special structure: it is a rank-one update of a diagonal matrix. The entries of the diagonal matrix are the negatives of the elasticity of intertemporal substitution (IES). The component $\boldsymbol{D}$ therefore corresponds to the direct effect of a price change. For illustration, assume that consumption preferences are CRRA, with $u(c) = \frac{1}{1-\gamma}c^{1-\gamma}$. This implies that $\gamma_t^* = \gamma$ is constant across periods, and the log Jacobian of the consumption function can be written as

$$\boldsymbol{\mathcal{C}} = -\frac{1}{\gamma}\boldsymbol{I} - \frac{1}{\gamma}\boldsymbol{1}\boldsymbol{\rho}' = -\frac{1}{\gamma} \times \begin{pmatrix} 1+\rho_1 & \rho_2 & \rho_3 & \rho_4 & \rho_5 & \cdots \\ \rho_1 & 1+\rho_2 & \rho_3 & \rho_4 & \cdots & \\ \rho_1 & \rho_2 & 1+\rho_3 & \cdots & & \\ \rho_1 & \rho_2 & \cdots & & & \\ \rho_1 & \cdots & & & & \\ \cdots & & & & & \end{pmatrix} \tag{28}$$

A change in price p_t at date t affects consumption at all dates k the same, namely by $-\frac{1}{\gamma}\rho_t$, except for the contemporaneous diagonal entry, which is $-\frac{1}{\gamma}(1 + \rho_t)$.

Leveraging Lemma 2, we now show that intertemporal export supply elasticities take a special form in this RA economy with permanent-income consumers. We illustrate the sequence-space Jacobian of $\mathcal{X}^*$ in Figure 1, which follows directly from Lemma 2 since

$$\boldsymbol{\mathcal{E}}_{kt} = \frac{\partial \log \mathcal{X}_k^*}{\partial \log p_t} = -\frac{c_k^*}{x_k^*}\frac{\partial \log \mathcal{C}_k^*}{\partial \log p_t} = \frac{1}{\gamma_k^*}\frac{c_k^*}{x_k^*} + \frac{1}{\gamma_k^*}\frac{c_k^*}{x_k^*}\frac{\partial \log \lambda^*}{\partial \log p_t} \quad \Longrightarrow \quad \boldsymbol{\mathcal{E}} = \mathrm{diag}(\hat{\boldsymbol{\phi}}) + \hat{\boldsymbol{\phi}}\boldsymbol{\rho}',$$

where we denote by $\hat{\boldsymbol{\phi}}$ the infinite column vector with entries $\frac{1}{\gamma_t^*}\frac{c_t^*}{x_t^*}$. It follows directly from Lemma 2, therefore, that the iXE matrix $\boldsymbol{\mathcal{E}}$ also takes the form of a rank-one shift of a diagonal matrix. Panel (a) plots the responses of Foreign exports to a price shock in the far future, $\frac{\partial \mathcal{X}_k^*}{\partial p_t}$, for $k \in [t-10, t+10]$. In other words, the figure plots part of the t-th column of $\boldsymbol{\mathcal{E}}$. Panel (b) is simply a zoomed-in version for additional clarity.[6] The figure graphically illustrates the stark shape of

[6] We evaluate the export supply function around $\tau_t = 0$ for illustration. This is the relevant object to ask in which

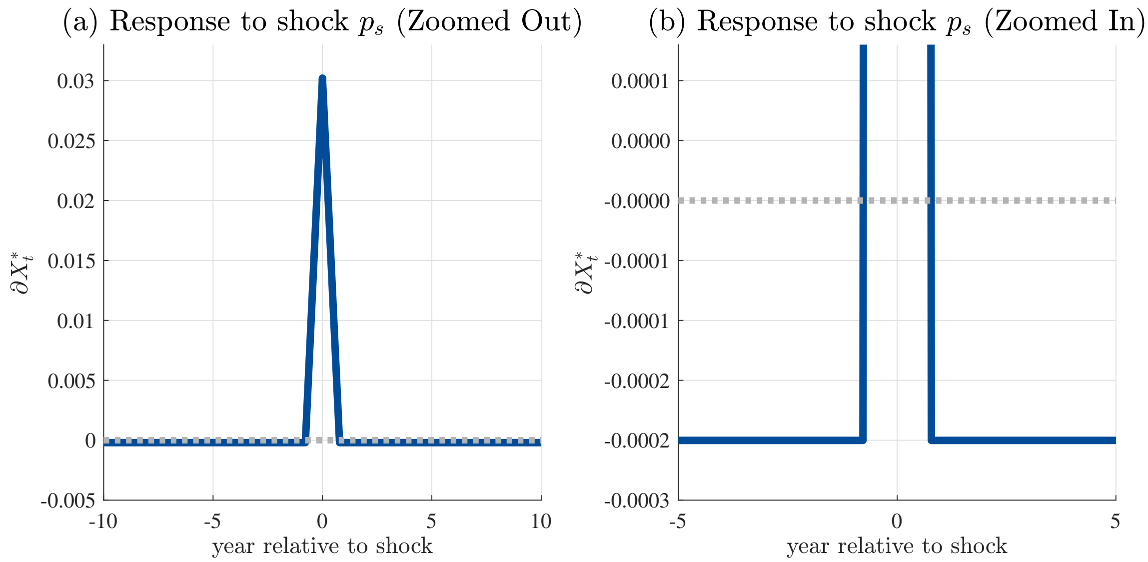

Figure 1. Column t of the sequence-space Jacobian of $\mathcal{X}^*$

this sequence-space Jacobian. Foreign export supply sharply rises contemporaneously with an increase in the intertemporal consumption price, but falls by the exact same amount at all other dates, leading to a slightly negative but completely flat profile at all dates $k \neq t$.

3.1.3 Optimal Tariffs: Revisiting Costinot et al. (2014)

We conclude this subsection by characterizing the optimal import tariff in this RA endowment economy. In particular, we show that the special structure of $\mathcal{X}^*$ and its sequence-space Jacobian implies that the intertemporal (cross-price elasticity) terms in our tariff formula (13) cancel out. This implies that the first-best optimal tariff formula becomes static in economies with permanent-income consumers.

This also allows us to relate our results to the seminal analysis of Costinot et al. (2014). They study the same RA endowment environment as we do in this subsection but take a primal approach to characterize optimal tariffs. Our discussion in this subsection clarifies how our dual approach relates to their primal approach, and how our optimal tariff formula simplifies to theirs in this special RA environment.

Primal efficiency condition. As a first step, we show that the efficiency condition (12) for Home's intertemporal terms of trade simplifies due to the special structure of the sequence-space Jacobian

direction the planner wants to push tariffs relative to the laissez fair competitive equilibrium. We set $y_t = y_t^* = 0.5$ for all t. Consumption preferences are CRRA with inverse IES $\gamma = 2$ and discount factor $\beta = 0.96$.

of $\mathcal{X}^*$. Using $\mathcal{X}_t^*(\boldsymbol{p}) = y_t^* - \mathcal{C}_t^*(\boldsymbol{p})$ and equation (26), condition (12) can be rewritten as

$$0 = \sum_k \beta^k u'(c_k) \frac{\partial \mathcal{X}_k^*}{\partial p_t} = \frac{\partial \lambda^*}{\partial p_t} \sum_k p_k \frac{u'(c_k)}{u''(c_k^*)} + \lambda^* \frac{u'(c_t)}{u''(c_t^*)}.$$

Crucially, the term $\sum_k p_k \frac{u'(c_k)}{u''(c_k^*)}$ is a constant that does not vary with t. We can now plug in for $\frac{\partial \lambda^*}{\partial p_t}$ using (27), rearrange, and arrive at

$$u'(c_t) = \frac{\sum_k p_k \frac{u'(c_k)}{u''(c_k^*)}}{\sum_k p_k} \left[u'(c_t^*) - u''(c_t^*) x_t^* \right]$$

This is precisely the efficiency condition Costinot et al. (2014) derive using the primal approach—equation (6) in their paper. The constant $-\sum_k p_k \frac{u'(c_k)}{u''(c_k^*)} / \sum_k p_k$ corresponds exactly to their Lagrange multiplier "μ". We thus showed that our efficiency condition in the dual representation collapses to that of Costinot et al. (2014) in this special environment. This derivation illustrates that our dual approach is closely related to the primal approach taken in other papers to characterize optimal tariff formulas.

Optimal tariffs. We now show that our intertemporal tariff formula also takes a special form in this economy with permanent-income consumers. We can use (26) to rewrite our intertemporal tariff formula (13) as

$$\frac{1}{\tau_t} = \sum_k \omega_{kt} \frac{\partial \log \mathcal{X}_k^*}{\partial \log p_t} = \sum_k \omega_{kt} \frac{p_t}{x_k^*} \frac{\partial \mathcal{X}_k^*}{\partial p_t} = -\sum_k \omega_{kt} \frac{p_t}{x_k^*} \frac{\partial \mathcal{C}_k^*}{\partial p_t} = -\frac{\partial \lambda^*}{\partial p_t} \sum_k \omega_{kt} \frac{p_t}{x_k^*} \frac{\partial \tilde{\mathcal{C}}_k^*}{\partial \lambda^*} - \frac{p_t}{x_t^*} \frac{\partial \tilde{\mathcal{C}}_t^*}{\partial p_t},$$

where we used $\omega_{tt} = 1$, or in matrix notation

$$\frac{1}{\tau_t} = \boldsymbol{\omega}_t \cdot \boldsymbol{\mathcal{E}}_t = \boldsymbol{\omega}_t \cdot \hat{\boldsymbol{D}}_t + \boldsymbol{\omega}_t \cdot \hat{\boldsymbol{P}}_t,$$

where $\hat{\boldsymbol{D}} = \text{diag}(\hat{\boldsymbol{\phi}})$ and $\hat{\boldsymbol{P}} = \hat{\boldsymbol{\phi}} \boldsymbol{\rho}'$. In words, the optimal tariff at date t is governed by two forces. Foreign's exports respond to a price change at date t directly, and at all dates k indirectly due to the effect on lifetime wealth. The next Proposition establishes that these indirect effects exactly cancel out in the determination of Home's optimal tariff. Our intertemporal tariff formula therefore becomes static.

Proposition 4. *In this RA endowment economy with permanent income consumers, we have*

$$\boldsymbol{\omega}_t \cdot \hat{\boldsymbol{P}}_t = 0\,, \qquad \textit{or equivalently} \qquad \frac{\partial \lambda^*}{\partial p_t} \sum_k \omega_{kt} \frac{p_t}{x_k^*} \frac{\partial \tilde{\mathcal{C}}_k^*}{\partial \lambda^*} = 0.$$

The intertemporal tariff formula (13) *therefore collapses to a static tariff formula given by*

$$\tau_t = \frac{1}{\boldsymbol{\mathcal{E}}_{tt} - \hat{\boldsymbol{P}}_{tt}} = -\frac{u''(c_t^*)}{u'(c_t^*)} x_t^*, \tag{29}$$

as in Costinot et al. (2014). The optimal tariff is the inverse of the contemporaneous export supply elasticity $\boldsymbol{\mathcal{E}}_{tt}$ *net of the contemporaneous indirect effect* $\hat{\boldsymbol{P}}_{tt}$*.*

Proposition 4 shows that our intertemporal tariff formula (13) collapses to a static one in this environment with permanent-income consumers. The optimal tariff τ_t is determined *as if* by the classic tariff formula (1) — using not the entire contemporaneous export supply elasticity $\boldsymbol{\mathcal{E}}_{tt}$, but instead netting out the indirect effect of the contemporaneous price change $\hat{\boldsymbol{P}}_{tt}$. Proposition 4 uses Lemma 2 to decompose the iXE matrix $\boldsymbol{\mathcal{E}}$ into the direct and indirect effects of price changes on export supply, and then shows that these indirect effects exactly offset each other in the determination of the optimal tariff τ_t. That is, $\boldsymbol{\omega}_t \cdot \hat{\boldsymbol{P}}_t = 0$. In this special case, our intertemporal tariff formula becomes static. And since we deliberately work in the exact same environment as Costinot et al. (2014) for illustration, we recover their tariff formula (29) exactly. The optimal tariff is determined *as if* by the classic tariff formula, but using a modified static export supply elasticity $\boldsymbol{\mathcal{E}}_{tt} - \hat{\boldsymbol{P}}_{tt}$ that nets out the indirect contemporaneous effect. In other words, the optimal tariff is governed by the diagonal component $\hat{\boldsymbol{D}}$ of the iXE matrix $\boldsymbol{\mathcal{E}}$.

Optimal tariffs are proportional to the trade balance. As already observed by Costinot et al. (2014), the optimal tariff takes a stark form in this representative agent endowment economy: it is proportional to the trade balance. Home internalizes the price impact that choosing one domestic allocation over another has through the world market clearing condition for the consumption good. When $x_t^* > 0$ and Home runs a trade deficit, it is effectively a net demander of the consumption good at date t. The tariff formula (29) implies that the optimal import tariff is positive in this case, $\tau_t > 0$, since $-\frac{u''(c_t^*)}{u'(c_t^*)} > 0$. By setting a positive import tax, the Home planner discourages households from importing the consumption good from Foreign, which in turn leads to a fall in the contemporaneous world price. And a lower price in turn implies that Home has to spend less on the (infra-marginal) units of the consumption good it is already importing from Foreign. This is precisely the sense in which Home internalizes its price impact. The optimal tariff therefore reflects Home's incentive to manipulate its intertemporal terms of trade as a monopolist in the market for intertemporal consumption claims. The surplus Home can extract from its trading partner depends on the sign and size of the trade balance, and so does the optimal tariff.

Optimal tariff dynamics. How does Home's optimal tariff evolve along a transition path of exogenous endowments $\boldsymbol{y}$ and $\boldsymbol{y}^*$? In Figure 2, we consider the case where Home initially has a larger endowment than Foreign, but the two endowments converge over time. This experiment

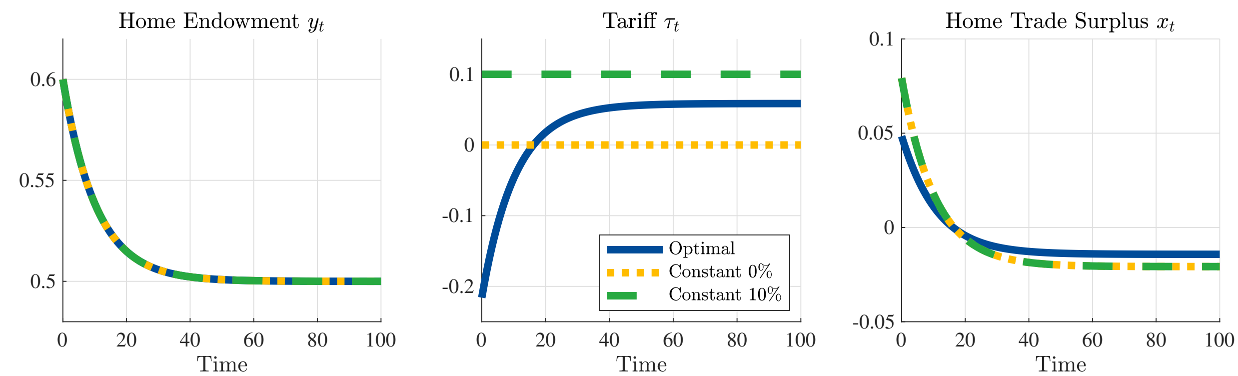

Figure 2. Tariff Dynamics

is intended to reflect a situation where the rest of the world is initially poorer but converges to Home gradually over the course of roughly 50 years. In this environment, Home's consumption-smoothing motive implies an initial trade surplus that eventually turns into a permanent deficit. The optimal tariff is proportional to the trade balance and therefore mirrors these dynamics. The Home planner initially taxes exports, $\tau_t < 0$, to raise the world price and extract a larger export revenue. Once the trade surplus turns into deficit and Home starts importing, it becomes optimal to start taxing imports to lower world prices and therefore total import expenditures.

Do tariffs affect the long-run trade balance? Recent policy debate has shown a spotlight on the question of whether tariffs can affect and help close the current trade deficit of the U.S. This debate has focused largely on the long-run tariff rather than transition dynamics. Figure 2 demonstrates that this focus is misguided in models of intertemporal trade and that it is important to consider the entire Ramsey plan, as we do here. We plot the economy's transition dynamics under constant 0% (green line) and 10% (yellow line) tariffs set once and for all at date 0. Both of these tariff regimes imply the exact same dynamics for the Home trade balance. Home initially runs a trade surplus but converges to a long-run trade deficit. Figure 2 illustrates that the level of the long-run tariff has no effect on the trade balance at all—its dynamics or its steady state level—as long as the tariff is set to be constant over time. This contrasts sharply with the optimal tariff regime, which initially subsidizes exports and later taxes imports. Under this regime, the Home trade balance converges to a smaller trade deficit in the long run. This illustrates that different tariff regimes do affect the long-run trade balance in dynamic environments and, crucially, that the optimal tariff regime implies a smaller long-run trade deficit.

3.2 Heterogeneous Agents and Incomplete Markets

As our next micro-foundation of intertemporal export supply elasticities, we model Foreign as a Beweley-Huggett-Aiyagari incomplete markets economy. Households face uninsurable income risk and they can only trade a riskfree bond. We abstract from capital and aggregate uncertainty for simplicity. The key insight that emerges from this application is that Foreign's export supply function $\mathcal{X}^*$ and its sequence-space Jacobian take a starkly different shape in incomplete markets environments. We focus our exposition on Foreign in this subsection since the details of the Home economy do not matter for the determination of $\mathcal{X}^*$.

3.2.1 Environment

Households. There is a continuum of households $i \in \mathcal{I}^* = [0,1]$ in Foreign. Household i's lifetime utility is defined by

$$V_0^i = \sum_t \beta^t \sum_{s^t} \pi(s^t) u^i(c_t^i(s^t)). \tag{30}$$

We abstract from a labor supply choice and assume that each household inelastically supplies 1 unit of labor.

Households can trade a one-period riskfree bond that represents a claim to one unit of the consumption good in the subsequent period. We denote the value of i's bond purchases at date t and history s^t in units of the consumption good by $a_{t+1}^i(s^t)$. The budget constraint is

$$a_{t+1}^i(s^t) + c_t^i(s^t) = w_t^* z_t^i(s^t) + R_t a_t^i(s^{t-1}), \tag{31}$$

where $R_t = 1 + r_t$ is the gross rate of return on savings. We denote by w_t^* the per-period local wage rate for effective labor. While each household supplies one unit hours inelastically, their individual labor productivity $z_t^i(s^t)$ varies and evolves stochastically over time, representing the source of idiosyncratic risk. Since we abstract from aggregate uncertainty, macroeconomic objects such as prices are not contingent on the realization of the history s^t. Finally, each household i faces a borrowing constraint of the form

$$a_{t+1}^i(s^t) \geq \underline{a}. \tag{32}$$

The problem of household i is therefore to maximize (30) subject to (31) and (32), taking as given the sequences of interest rates $\boldsymbol{R}$ and wages $\boldsymbol{w}^*$. It is useful to switch to a recursive representation of the household problem. We can uniquely associate households with their idiosyncratic state variables (a, z), where a is the household's beginning-of-period wealth. Dropping i superscripts, the policy functions for savings and consumption of households in Foreign are then given by $a_{t+1}^*(a, z)$ and $c_t^*(a, z)$. Finally, we denote the joint density of households over wealth and labor productivities by $g_t^*(a, z)$ and its law of motion is characterized by the usual Kolmogorov forward equation.

Firms. A representative and perfectly competitive firm produces the final consumption good using technology $y_t^* = A_t^* \ell_t^*$, where ℓ_t^* denotes total use of effective labor. Profit maximization implies the optimality condition

$$w_t^* = A_t^*.$$

Markets and equilibrium. Three markets must clear in each period. Labor market clearing requires

$$\ell_t^* = \iint z g_t^*(a,z)\, da\, dz.$$

The world goods market clears when total Foreign exports are equal to Home imports,

$$x_t^* \equiv A_t^* \ell_t^* - \iint c_t^*(a,z) g_t^*(a,z)\, da\, dz = c_t - y_t,$$

where y_t and c_t denote aggregate output and consumption of Home. Finally, we assume that the bond is in zero net supply globally, so that one country's net foreign asset position must be equal to the other country's liabilities. We drop the asset market clearing condition by Walras' law.

Definition 2 (Competitive Equilibrium in Foreign)**.** *Given an initial density $g_0^*(a,z)$, a Home allocation $(\boldsymbol{c}, \boldsymbol{y})$ and a technology sequence $\boldsymbol{A}^*$, competitive equilibrium in Foreign comprises an aggregate allocation $(\boldsymbol{c}^*, \boldsymbol{\ell}^*, \boldsymbol{y}^*)$, prices $(\boldsymbol{R}, \boldsymbol{w}^*)$, policy functions $\{c_t^*(a,z)\}$ and joint densities $\{g_t^*(a,z)$ so that (i) households optimize, (ii) firms optimize, (iii) markets clear, and (iv) the evolution of the joint density is consistent with household behavior.*

To relate our results to Sections 2 and 3.1, it will be convenient to represent equilibrium conditions in terms of intertemporal consumption prices p_t instead of per-period rates of return R_t. Since R_0 is not determined as part of equilibrium, we set it to 1. We can then define the intertemporal price of consumption at date t as

$$p_t = \prod_{s=0}^{t} \frac{1}{R_s}. \tag{33}$$

Since our normalizations imply $p_0 = \frac{1}{R_0} = 1$, we treat consumption at date 0 as our numeraire, so that p_t denotes the intertemporal price of date t consumption relative to date 0 consumption.

3.2.2 Intertemporal Export Supply Elasticities

Our analysis in Section 2 assumes the existence of an intertemporal export supply function $\mathcal{X}^*$ that satisfies the external balance condition in (9). We now show how to construct this function in this HA economy and characterize its properties.

Foreign exports are aggregated from household consumption behavior at the micro level according to $x_t^* = A_t^* \iint z g_t^*(a,z)\, da\, dz - \iint c_t^*(a,z) g_t^*(a,z)\, da\, dz$. Households' behavior at date t depends on their current state variables and on the path of future prices from t onwards. At the same time, date t state variables are determined by initial conditions at date 0 and consumption-savings decisions between dates 0 and t, which in turn depend on prices. Consequently, households' policy functions at date t can be expressed entirely in terms of the paths of prices starting from date 0 as shown in Auclert et al. (2024b). Using firm optimality to determine wages as $w_t^* = A_t^*$, Foreign date t aggregate consumption c_t^* can therefore be expressed in terms of an intertemporal consumption function $\mathcal{C}^*$, given by

$$c_t^* = \iint c_t^*(a,z) g_t^*(a,z)\, da\, dz = \mathcal{C}_t^*(\boldsymbol{p}).$$

Therefore, the intertemporal export supply function is simply

$$\mathcal{X}_t^*(\boldsymbol{p}) = A_t^* - \mathcal{C}_t^*(\boldsymbol{p}).$$

We still have to show that $\mathcal{X}^*$ satisfies the external balance condition (9). Starting from households' budget constraints (31) and aggregating yields

$$NFA_{t+1}^* = R_t NFA_t^* + w_t^* \ell_t^* - c_t^*,$$

where we define $NFA_t^* = \iint a g_t^*(a,z)\, da\, dz$. Using $w_t^* = A_t^*$ and $x_t^* = y_t^* - c_t^*$, iterating forward yields

$$NFA_0^* = -\sum_t p_t \mathcal{X}_t^*(\boldsymbol{p}),$$

where NFA_0^* is the value of Foreign's initial asset position and where we used $\lim_{T\to\infty} p_T NFA_T^* = 0$, which follows directly from household optimality at the micro level. In other words, the intertemporal export supply function $\mathcal{X}^*$ satisfies this aggregate external balance condition because, in response to a change in prices $\boldsymbol{p}$, the consumption-savings behavior of Foreign households must still satisfy their individual budget constraints. And the above condition follows simply from aggregating individual budget constraints.

Just like in the RA model, the intertemporal export supply function captures everything the Home planner has to know about Foreign. Unlike in the RA model, however, the micro-foundation of $\mathcal{X}^*$ in HA captures the rich dynamics of household heterogeneity in Foreign. It encodes how the entire income and wealth distribution of Foreign reacts to changes in the price sequence $\boldsymbol{p}$ and therefore tariff policy.

In the presence of household heterogeneity and incomplete markets, the sequence-space Jacobian of $\mathcal{X}^*$ has a starkly different shape than in the RA environment of Section 3.1. We illustrate this in Figure 3, which compares the t-th column of the Jacobian matrix $\boldsymbol{\mathcal{E}}^*$ in the RA and HA models of Sections 3.1 and 3.2. Panel (a) overlays the columns of both matrices, whereas Panels (b) and (c) present zoomed-in versions for each model to make the comparison easier.

What the two Jacobians have in common is that a price increase at date t increases Foreign exports contemporaneously at date t but decreases exports at all other dates $k \neq t$. Beyond this qualitative similarity, Figure 3 displays three sharp differences between the two models.

First, and most importantly, the profile of the export supply response in HA at dates $k \neq t$ is not like it is in RA. In RA, we already observed in Section 3.1 that the response profile at dates $k \neq t$ is entirely flat, as illustrated again in Panel (b). In the HA economy with incomplete markets, households no longer behave like the infinitely-forward looking permanent-income consumers

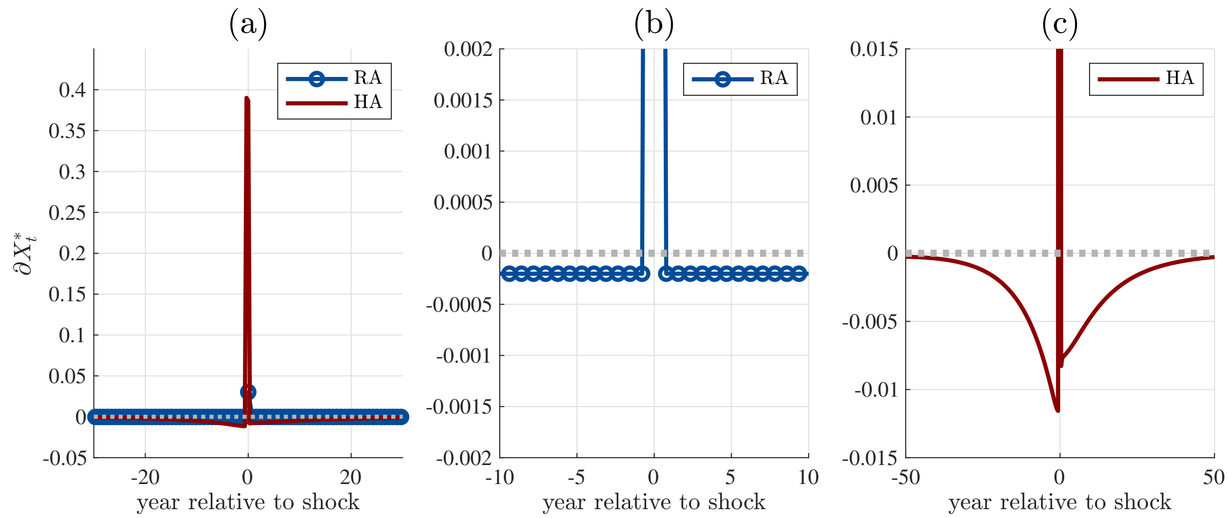

Figure 3. Column t of the sequence-space Jacobian of $\mathcal{X}^*$ in RA and HA

of Section 3.1. Instead, households' effective planning horizon is much shorter; households make forward-looking plans until they hit their borrowing constraint. The strength of anticipation effects governing a household's behavior is therefore determined by that household's distance to the borrowing constraint—the behavior of households with very large a is still similar to that of permanent-income consumers but households with low a close to the borrowing constraint $\underline{a}$ behave increasingly hand-to-mouth. Therefore, in response to an anticipated price increase at date t, households do not respond at all for k more than 50 years prior to t because almost all households expect to hit the borrowing constraint before reaching period t. This can be seen in Panel (c), where the export supply response dissipates entirely for small k.

As with the anticipation effect, the persistence of the export supply response to the price increase is also different. In the RA model, households continue to consume the annuity value of the change in lifetime wealth after date t, just like they did in anticipation before date t. In the HA model, on the other hand, households save initially but start dissaving as they draw negative earnings shocks and get closer to their borrowing constraints. As all households eventually hit their borrowing constraints, at which point they have fully dissaved the initial effect of the price change, the export supply response eventually fully dissipates for large enough k.

Second, conditional on responding to the price change at dates k before or after t, households in HA respond more strongly. This can be seen in Figure 3 by noting that the scale of the negative effect in Panel (c) is almost two orders of magnitude larger for k close to t than it is in Panel (b). This is because households have much larger marginal propensities to consume (MPC) in HA than they do in RA, and consumption responses to price changes are partly governed by and proportional to MPCs.

Third, and relatedly to the previous point, the contemporaneous positive response of exports to the price increase at date t is one order of magnitude larger in HA than it is in RA. This can

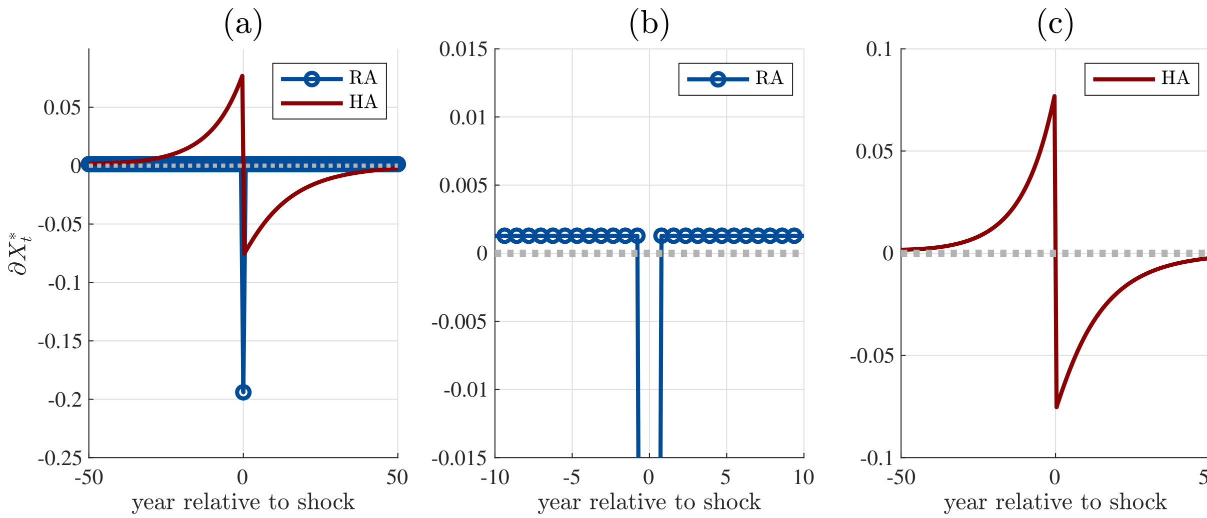

Figure 4. Column t of the sequence-space Jacobian of $\mathcal{X}_r^*$ in RA and HA

be seen in Panel (a). The contemporaneous response at date t is roughly 0.4 in HA, whereas it is exactly 0.04 in RA (the blue dot at $k - t = 0$).

In summary, the export supply elasticities that govern the intertemporal terms of trade manipulation of optimal tariff policy differ starkly across the RA and HA models of Sections 3.1 and 3.2. While the off-diagonal entries of the iXE matrix $\boldsymbol{\mathcal{E}}^*$ are constant in the RA model, implying anticipation in and persistence of the export supply response that are infinitely lived, the off-diagonal entries in the HA model are no longer constant and decline in their distance to the diagonal. In other words, the substitution elasticities between different periods are constant in the RA model but not in the HA model. With incomplete markets, there is a natural notion of "distance" to an announced price change.

Export supply elasticities to interest rates. We started with a discussion of export supply elasticities to intertemporal prices p_t to facilitate comparison to Sections 2 and 3.1. Intertemporal prices in these settings are functions of per-period interest rates, according to equation (33). In Figure 4, we also plot the sequence-space Jacobian of the intertemporal export supply function $\mathcal{X}_r^*(\boldsymbol{r})$ expressed in terms of sequences of net interest rates. This is useful in part for comparability to the heterogeneous agent macro literature, which often reports sequence-space Jacobians in terms of interest rates.

4 Multiple Goods

This section introduces multiple goods, allowing for both intra- and intertemporal trade. We assume that all goods appear as exogenous endowments in both countries. This allows us to abstract for now from the issue of non-differentiability that emerges in Ricardian models of trade with comparative advantage in production (Costinot et al., 2015). We view this as orthogonal to the observations we make in this section.

4.1 Preferences, Endowments and Implementability

There are two countries, and time is discrete as before. In each period t, there is a finite number J of tradable consumption goods which we denote by j and k. We abstract from household heterogeneity and uncertainty in this section for simplicity, but bring both of these features back in our quantitative model in Section 5.

Preferences. There is a representative household in each country whose preferences are

$$V_0 = \sum_{t \geq 0} \beta^t u(c_t) \qquad \text{and} \qquad V_0^* = \sum_{t \geq 0} \beta^t u(c_t^*)$$

where c_t and c_t^* denote the consumption bundles of Home and Foreign. These are given by

$$c_t = \mathcal{D}(\{c_{jt}\}_j) \qquad \text{and} \qquad c_t^* = \mathcal{D}^*(\{c_{jt}^*\}_j).$$

We make no assumptions about the homotheticity of $\mathcal{D}(\cdot)$ and $\mathcal{D}^*(\cdot)$.

Endowments. Each good j appears as an exogenous endowment in both countries, denoted y_{jt} and y_{jt}^*. We assume that $y_{jt} > 0$ and $y_{jt}^* > 0$ for all j and t. We also assume as before that the infinite sequences $\boldsymbol{y}_j = \{y_{jt}\}_{t \geq 0}$ and $\boldsymbol{y}_j^* = \{y_{jt}^*\}_{t \geq 0}$ are deterministic and convergent for all j, so agents have perfect foresight over them.

Resource constraints. The resource constraint for consumption good j at date t is now given by

$$x_{jt}^* = c_{jt} - y_{jt},$$

where x_{jt}^* denotes Foreign's aggregate export supply.

Intertemporal export supply function and implementability. We denote the intertemporal price of good j at date t by p_{jt}. As before, we use bold-faced notation for infinite sequences, denoting by $\boldsymbol{p}_j = \{p_{jt}\}_{t \geq 0}$. It will also be useful to collect all price sequences $\boldsymbol{p} = \{\boldsymbol{p}_j\}_j$. Similarly, we define $\boldsymbol{c}_j = \{c_{jt}\}_{t \geq 0}$ and $\boldsymbol{c} = \{\boldsymbol{c}_j\}_j$. As in Section 2.2, we start by assuming the existence of an intertemporal export supply function $\mathcal{X}^*$ that is differentiable and satisfies

$$x_{jt}^* = \mathcal{X}_{jt}^*(\boldsymbol{p}) = y_{jt}^* - \mathcal{C}_{jt}^*(\boldsymbol{p}) \qquad \text{and} \qquad NFA_0^* + \sum_t \sum_j p_{jt} \mathcal{X}_{jt}^*(\boldsymbol{p}) = 0. \tag{34}$$

The intertemporal export supply function maps the infinite price sequences $\boldsymbol{p}$ for all goods to a level of Foreign export supply for good j at each date t. We again assume that $\mathcal{X}^*$ satisfies an aggregate external balance condition, so that Foreign's initial NFA_0^* is equal to the sum of future trade balances. It is also useful to explicitly assume the existence of the Foreign consumption function $\mathcal{C}^*$, which maps price sequences to a level of consumption.

4.2 Home Efficiency and First-Best Tariff

We now follow Sections 2.3 – 2.4 and characterize efficient allocations from the perspective of Home, as well as the tariffs that help decentralize them.

We start with a statement of implementability that mirrors Lemma 1 in Section 2.2: Taking as given endowment sequences $\boldsymbol{y}$ and $\boldsymbol{y}^*$, a Home allocation $\boldsymbol{c}$ and prices $\boldsymbol{p}$ are implementable if and only if they satisfy

$$\mathcal{X}_{jt}^*(\boldsymbol{p}) = c_{jt} - y_{jt}$$

for all j and t. Efficient allocations therefore maximize the lifetime utility of Home's representative household subject to the implementability condition. The associated Lagrangian is given by

$$L = \sum_t \beta^t u(\mathcal{D}(c_{jt})) + \sum_t \sum_j \mu_{jt} \left[\mathcal{X}_{jt}^*(\boldsymbol{p}) - c_{jt} + y_{jt} \right],$$

with first-order conditions $\beta^t u'(c_t) \frac{\partial \mathcal{D}}{\partial c_{jt}} = \mu_{jt}$ and $0 = \sum_s \sum_k \mu_{ks} \frac{\partial \mathcal{X}_{ks}^*}{\partial p_{jt}}$ for all t and j. Putting these together, we get

$$0 = \sum_s \beta^s u'(c_s) \sum_k \frac{\partial \mathcal{D}}{\partial c_{ks}} \frac{\partial \mathcal{X}_{ks}^*}{\partial p_{jt}}, \tag{35}$$

which is the analog to the efficiency condition (12) for Home's intertemporal terms of trade in the one-good environment.

Optimal tariff. We now characterize the optimal tariff that helps decentralize efficiency condition (35). In particular, we allow for import tariffs τ_{jt} that vary across goods and periods. Since we assumed Foreign's export supply function to satisfy the external balance condition (34) for all positive price sequences, we can differentiate and obtain

$$0 = \sum_s \sum_k p_{ks} \frac{\partial \mathcal{X}_{ks}^*}{\partial p_{jt}} + x_{jt}^*.$$

Next, if tariffs τ_{jt} decentralize the Home efficiency allocation in a world competitive equilibrium, then it must be the case that

$$\beta^{t-s}\frac{u'(c_t)}{u'(c_s)}\frac{\partial\mathcal{D}/\partial c_{jt}}{\partial\mathcal{D}/\partial c_{ks}} = \frac{(1+\tau_{jt})p_{jt}}{(1+\tau_{ks})p_{ks}}.$$

We plug this condition back into Home's efficiency condition (35) and obtain

$$0 = \sum_s\sum_k\left(\beta^t u'(c_t)\frac{\partial\mathcal{D}}{\partial c_{jt}}\right)\frac{(1+\tau_{ks})p_{ks}}{(1+\tau_{jt})p_{jt}}\frac{\partial\mathcal{X}^*_{ks}}{\partial p_{jt}} = \sum_s\sum_k p_{ks}\frac{\partial\mathcal{X}^*_{ks}}{\partial p_{jt}} + \sum_s\sum_k \tau_{ks}p_{ks}\frac{\partial\mathcal{X}^*_{ks}}{\partial p_{jt}}$$

And finally we put the previous two equations together, multiply and divide by τ_{jt}, and rearrange. This yields the optimal tariff formula for environments with multiple goods.

Proposition 5 (Intertemporal Tariff Formula: Multiple Goods)**.** *As part of a decentralization of the Home efficiency allocation, the efficiency condition* (35) *for intertemporal terms of trade is satisfied if Home consumers face an ad-valorem import tariff given by*

$$\frac{1}{\tau_{jt}} = \sum_s\sum_k \omega^{ks}_{jt}\frac{\partial\log\mathcal{X}^*_{ks}}{\partial\log p_{jt}}$$

where ω^{ks}_{jt} *denotes a relative tariff revenue weight defined as*

$$\omega^{ks}_{jt} = \frac{\tau_{ks}p_{ks}x^*_{ks}}{\tau_{jt}p_{jt}x^*_{jt}}.$$

5 A HANK Model with Trade

In this section, we develop a quantitative heterogeneous agent New Keynesian (HANK) model with multi-sector trade. Time is discrete and we abstract from aggregate uncertainty. Our model features heterogeneous households and multiple goods. Home and Foreign are each populated by a measure one continuum of households who face uninsurable idiosyncratic risk. There are J types of goods that come in Home- and Foreign-produced varieties. In other words, there are J production sectors in each country, with each sector producing a unique variety. Households in both countries have homothetic CES preferences over these $2\times J$ varieties.

Section 5.1 presents the key model elements and defines competitive equilibrium. Our exposition focuses on Home for simplicity since the environment in Foreign is symmetric. We set up the model with a rich set of instruments, which allows us to compare different policy regimes in our numerical analysis. Section 5.2 takes the model to the data. The calibrated model matches both trade elasticities and key moments of the income and wealth distribution. Our numerical analysis starts with a positive exploration of the consequences of tariff shocks in Section 5.3. We then compute the Ramsey steady state of our model in Section 5.4 and show that the optimal long-run import tariff it implies is 5.4%. Finally, we analyze optimal tariff dynamics in response to TFP shocks from a timeless perspective. Our numerical experiments presently focus on the benchmark where monetary policy restores the flexible price allocation in both countries, as we discuss below. This is a useful benchmark to study optimal tariff policy because it implies that Ramsey optimal second-best tariffs are not used to close labor wedges. In ongoing work, we also consider the case where monetary policy instead follows a Taylor rule and therefore fails to restore production efficiency. In that case, using tariffs for the purpose of terms of trade manipulation comes at the additional cost of creating labor wedges. This section is accompanied by an online appendix (Appendix D) that provides a detailed statement of the optimal tariff Ramsey problem and derives the key optimality conditions that characterize the Ramsey plan.

5.1 Model

5.1.1 Households

Preferences. The preferences of a household in Home are defined as

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \Big[u(c_t) - v(\ell_t) \Big]$$

where c_t is a homothetic consumption aggregator of all varieties and ℓ_t denotes hours of work. We assume that the household's consumption aggregator is Cobb Douglas over the J types of goods and CES over varieties within each type of good,

$$c_t = \sum_j \alpha_j \log c_{j,t} \qquad \text{and} \qquad c_{j,t} = \left((1-\theta_j)^{\frac{1}{\eta_j}} c_{jH,t}^{\frac{\eta_j - 1}{\eta_j}} + \theta_j^{\frac{1}{\eta_j}} c_{jF,t}^{\frac{\eta_j - 1}{\eta_j}} \right)^{\frac{\eta_j}{\eta_j - 1}}. \tag{36}$$

We denote by $c_{j\omega,t}$ and $c^*_{j\omega,t}$ the consumption of a household in Home and Foreign, respectively, of good j variety ω. The Home- and Foreign-produced varieties are indexed by $\omega \in \{H, F\}$. The consumption preferences of households in Home and Foreign take the same nested CES form but we allow for different parameters for Foreign, denoted by $(\alpha^*_j, \theta^*_j, \eta^*_j)$. Calibrated differences between θ_j and θ^*_j allow us to capture home bias in trade. This is isomorphic to iceberg trade costs, which we consequently abstract from.

Budget constraint. We denote the world price of good j variety ω at date t by $p_{j\omega,t}$. The total consumption expenditure of a household in Home is therefore given by $\sum_j \sum_\omega (1 + \tau_{j\omega,t}) p_{j\omega,t} c_{j\omega,t}$, where $\tau_{j\omega,t}$ denotes Home's ad-valorem import tariff as in previous sections. Each country issues a nominal bond and we abstract from segmentation in international financial markets, so all households can trade the two bonds. A household in Home therefore faces the following budget constraint in units of domestic currency

$$a_{H,t+1} + \mathcal{E}_t a_{F,t+1} + \sum_j \sum_\omega (1+\tau_{j\omega,t}) p_{j\omega,t} c_{j\omega,t} = (1+i_t) a_{H,t} + \mathcal{E}_t (1+i_t^*) a_{F,t} + E_t.$$

We denote by $a_{H,t}$ and $a_{F,t}$ the value of the household's beginning-of-period holdings of the Home and Foreign bonds in units of local currency. And we denote by $\mathcal{E}_t$ the nominal exchange rate, i.e., the relative price of Foreign to Home currency. Finally, i_t and i_t^* denote the nominal rates of return on the two bonds, and E_t is the household's post-tax non-financial income. This comprises after-tax labor income, dividend income, and government transfers,

$$E_t = W_t z_t \ell_t + z_t \Pi_t + T_t(z_t),$$

where W_t is the nominal wage, $T_t(z_t)$ is a lump-sum rebate that may vary across households, and z_t denotes the household's individual labor productivity. It follows a first-order Markov chain with mean 1 and is the source of idiosyncratic risk. Labor supply decisions ℓ_t are intermediated by labor unions as we describe below and therefore taken as given by the household.

Household problem. Since the consumption aggregator is homothetic, there exists an ideal price index P_t that satisfies $P_t c_t = \sum_j \sum_\omega (1+\tau_{j\omega,t}) p_{j\omega,t} c_{j\omega,t}$. This assumption allows us to derive a recursive representation of the household problem in terms of total real wealth, defined as $a_t = \frac{a_{H,t} + \mathcal{E}_{t-1} a_{F,t}}{P_t}$. Since all households can trade the two bonds freely and there is no aggregate risk, the usual no-arbitrage condition must hold and implies uncovered interest parity (UIP),

$$1 + i_{t+1} = \frac{\mathcal{E}_{t+1}}{\mathcal{E}_t}(1 + i_{t+1}^*).$$

Since the household's portfolio positions in the two bonds are indeterminate in this setting, we write the household problem directly in terms of total real wealth. We can therefore uniquely associate each household with her state variables at date t and arrive at the following recursive representation of the household problem: A household with real wealth a and individual productivity z at date t solves the dynamic problem

$$\begin{aligned} V_t(a,z) \;=\; & \max_{\{c_{j\omega}\},\, a'} \; u\left(\left\{c_{j\omega}\right\}_{j\omega}\right) - v(\ell_t) + \beta \mathbb{E}_t\left[V_{t+1}(a', z')\right] \\ & \text{s.t} \quad a' = R_t a + e_t(z) - \sum_j \sum_\omega \frac{(1+\tau_{j\omega,t}) p_{j\omega,t}}{P_t} c_{j\omega} \\ & \qquad\;\; a' \geq \underline{a} \end{aligned}$$

taking as given the sequences of union-intermediated hours of work ℓ_t, real interest rates $R_t = (1+i_t)\frac{P_{t-1}}{P_t}$, real wages $w_t = \frac{W_t}{P_t}$, and cum-tariff prices $(1+\tau_{j\omega,t}) p_{j\omega,t}$. We denote by $e_t(z) = E_t(z)/P_t$

the household's real post-tax non-financial income, which may depend on the household's income state z because we allow transfers to be household-specific. Finally, households also face a constraint on borrowing in terms of real wealth, given by $a' \geq \underline{a}$.

The consumption-savings problem of households in Foreign admits the same recursive representation except that all variables are denoted with asterisks. For example, the real interest rate R_t^* faced by households in Foreign differs from that in Home because households have different homothetic consumption aggregators and thus different ideal price indices, even though rates of return on assets are equalized.

Cross-sectional distribution. We denote the joint density of households over real wealth and labor productivity at date t by $g_t(a,z)$. This joint density evolves according to the usual Kolmogorov forward equation, which we present in Appendix D.2. We also refer to the density $g_t(a,z)$ as Home's cross-sectional income and wealth distribution.

Labor unions. Household labor supply decisions are intermediated by labor unions as in Erceg et al. (2000) and Auclert et al. (2024b). We allow for flexible nominal wage adjustments but maintain the standard assumption of labor rationing: all households work the same hours, so ℓ_t does not depend on the state (a,z) of the household. Appendix D.4 presents a self-contained treatment of our model's labor market structure and shows that it gives rise to the aggregate labor supply schedule

$$v'(\ell_t) = \frac{\epsilon^w - 1}{\epsilon^w} w_t u'(C_t) \tag{37}$$

where ϵ^w denotes the elasticity of substitution that governs the labor union's desired markup of real wages over the marginal rate of substitution. It is a measure of monopsony in the labor market.

5.1.2 Multi-Sector Production

Production in each country takes place in J sectors. Each sector j produces the country-specific variety of good j. To keep notation symmetric, we denote by $y_{j\omega,t}$ and $y^*_{j\omega,t}$ the date t output of good j variety ω in Home and Foreign, respectively. But since Home varieties can only be produced in Home, and vice versa for Foreign, we have $y_{jF,t} = y^*_{jH,t} = 0$.

In the New Keynesian tradition, a production sector j comprises a retailer and a continuum of intermediate input firms whose dynamic pricing problem gives rise to sectoral New Keynesian Phillips curves. These sectoral intermediate inputs are non-tradable and cannot be used in production outside of sector j. They are consequently not relevant for trade or any other part of the model. Their only role is as a micro-foundation of sectoral Phillips curves. We discuss each production stage in turn.

Retailer. In each Home sector j, a retailer produces final good j by bundling a measure one continuum of sector-specific intermediate input varieties k, according to

$$y_{jH,t} = \left(\int_0^1 y_{jH,t}(k)^{\frac{\epsilon_j - 1}{\epsilon_j}} dk \right)^{\frac{\epsilon_j}{\epsilon_j - 1}},$$

with elasticity of substitution ϵ_j. CES aggregation implies the usual demand function for variety k,

$$y_{jH,t}(k) = \left(\frac{p_{jH,t}(k)}{p_{jH,t}} \right)^{-\epsilon_j} y_{jH,t},$$

where $p_{jH,t}(k)$ is the price of input k in Home sector j and $p_{jH,t}$ is the price of Home-produced final good j.

Intermediate firms. Intermediate input k in Home sector j is produced by a monopolistically competitive firm according to the production technology

$$y_{jH,t}(k) = A_{jH,t} \ell_{jH,t}(k).$$

The Hicks-neutral productivity shifter $A_{jH,t}$ is specific to each good and variety, which allows us to capture comparative advantage. We assume that $A_{jF,t} = A^*_{jH,t} = 0$, which is the technological analog of assuming that there are unique Home- and Foreign-produced varieties. Profits are equal to revenue net of salary payments, given by

$$\Pi_{jH,t}(k) = p_{jH,t}(k) y_{jH,t}(k) - (1 - \tau^f_{jH,t}) W_t \ell_{jH,t}(k).$$

We allow for an employment subsidy $\tau^f_{jH,t}$ that is specific to each sector and may vary over time. Real marginal costs therefore differ at the sector level but are common to all intermediate input producers,

$$mc_{jH,t} = \frac{(1 - \tau^f_{jH,t}) W_t}{A_{jH,t}}. \tag{38}$$

Firm k in sector j faces a quadratic Rotemberg adjustment cost when it changes its sales price $p_{jH,t}(k)$. This adjustment cost is given by $\frac{\chi_j}{2}(\frac{p_{jH,t}(k)}{p_{jH,t-1}(k)} - 1)^2 p_{jH,t} y_{jH,t}$, where χ_j governs the degree of price stickiness and may vary across production sectors. The firm's dynamic pricing problem is therefore to maximize the net present value of profits net of adjustment costs and subject to the demand it faces from the retailer. We define and characterize this problem in Appendix D.5.

Sectoral aggregation and Phillips curves. Under our assumption of Rotemberg adjustment costs, all intermediate input firms k in sector j will be symmetric ex post. In other words, as long as firms k and k' are initialized with the same prices $p_{jH,-1}(k) = p_{jH,-1}(k')$, they will choose the same prices and production plans from time 0 onwards. Symmetry within sectors therefore implies that $y_{jH,t}(k) = y_{jH,t}$ and $\ell_{jH,t}(k) = \ell_{jH,t}$ for all k and we can directly work with the sectoral production function

$$y_{jH,t} = A_{jH,t}\ell_{jH,t}. \tag{39}$$

Symmetry also implies that all firms k choose the same inflation rates, so that $p_{jH,t}(k) = p_{jH,t}$. We show in Appendix D.5 that the solution to the firm's dynamic pricing problem in a symmetric equilibrium therefore gives rise to a set of sectoral New Keynesian Phillips curves,

$$\pi_{jH,t} = \beta\pi_{jH,t+1} + \frac{\epsilon_j}{\chi_j}\left(mc_{jH,t} - \frac{\epsilon_j - 1}{\epsilon_j}\right), \tag{40}$$

that characterize the price dynamics in sector j, where we define sectoral price inflation as $\pi_{jH,t} = \frac{p_{jH,t}}{p_{jH,t-1}} - 1$. While deriving these sectoral Phillips curves in discrete time requires linearization, equation (40) is exact in continuous time.

5.1.3 Government Policy

Fiscal policy. Fiscal policy in Home comprises four sets of instruments: ad-valorem import tariffs $\tau_{j\omega,t}$, an employment subsidy on the labor union τ^u (see Appendix D.4), employment subsidies on firms $\tau^f_{jH,t}$, and the lump-sum transfer to households $T_t(z)$. The import tariff only applies to Foreign-produced good, so $\tau_{jH,t} = 0$ for all j and t. In our numerical analysis, we presently focus on the case of a uniform tariff $\tau_{jF,t} = \tau_t$ that applies equally to all imported goods. We abstract from government spending and deficit finance and assume that the fiscal authority runs a balanced budget in each period t. This requires that tariff revenue is equal to aggregate transfer payments plus subsidies,

$$\tau_t \sum_j p_{jF,t} \iint c_{jF,t}(a,z)g_t(a,z)\, da\, dz = \iint T_t(z)g_t(a,z)\, da\, dz + \sum_j \tau^f_{jH,t}W_t\ell_{jH,t} + \tau^u W_t\ell_t. \tag{41}$$

The fiscal authority in Foreign also sets employment subsidies and finances these using a lump-sum tax on households. But we assume that Foreign does not engage in tariff policy.

Monetary policy. Each country's central bank is tasked with setting nominal interest rates, i_t and i^*_t. We consider two alternative monetary policy regimes in our numerical experiments. Our main analysis in Sections 5.3 and 5.4 presently assumes that monetary policy implements the flexible-price allocation in each country—with the help of fiscal policy setting appropriate sectoral employment subsidies. Intuitively, monetary policy is set so as to close the aggregate labor wedge while fiscal policy sets time-varying subsidies $\tau^f_{jH,t}$ to align marginal costs across sectors. This regime is a useful benchmark to study optimal tariff policy because it implies that Ramsey optimal second-best tariffs are not used to close labor wedges.

In ongoing work, we study optimal tariffs under the alternative monetary policy regime where each central bank follows a Taylor rule. Home's policy rule is given by

$$1 + i_t = (1 + r_{ss}) \left(\frac{P_t}{P_{t-1}} \right)^{\phi_\pi} \left(\frac{Y_t}{Y_{ss}} \right)^{\phi_y}, \tag{42}$$

where r_{ss} denotes the steady state real interest rate. The central bank sets the nominal interest rate as a function of CPI inflation and the output gap, where $Y_t = \frac{\sum_j p_{jH,t} y_{jH,t}}{P_t}$ denotes real gross domestic product. Foreign's policy rule is symmetric. Under this policy regime, monetary policy does not restore the flexible price allocation. On the one hand, this implies that second-best tariff policy may be used for production efficiency gains in response to shocks. On the other hand, using tariffs for the purpose of terms of trade manipulation comes at the additional cost of creating labor wedges when monetary policy is not set optimally to restore production efficiency.

5.1.4 Markets and Equilibrium

The markets for goods, labor and bonds must clear in equilibrium. Goods market clearing requires that total output of good j variety ω is equal to total consumption in both countries,

$$y_{j\omega,t} + y^*_{j\omega,t} = \iint c_{j\omega,t}(a,z) g_t(a,z)\, da\, dz + \iint c^*_{j\omega,t}(a,z) g^*_t(a,z)\, da\, dz, \tag{43}$$

where of course $y_{jF,t} = y^*_{jH,t} = 0$ for all j. Labor market clearing requires that total supply of effective labor is equal to total use in production in each country,

$$\iint \ell_t z g_t(a,z)\, da\, dz = \sum_j \ell_{jH,t}, \tag{44}$$

and symmetrically for Foreign. Finally, we assume that both bonds are in zero net supply, so bond market clearing requires that households' total bond holdings aggregate to 0. Since portfolio allocation is indeterminate, we focus directly on total real wealth and Home's net foreign asset position, defined by $NFA_t = \iint a g_t(a,z)\, da\, dz$, where we must have

$$NFA_t + NFA^*_t = 0. \tag{45}$$

We can now define world competitive equilibrium taking as given fiscal and monetary policy in both countries.

Definition 3 (Competitive Equilibrium)**.** *Taking as given initial household distributions $g_0(a,z)$ and $g^*_0(a,z)$, shocks $(\boldsymbol{A}, \boldsymbol{A}^*)$ as well as policy $(\boldsymbol{i}, \boldsymbol{i}^*, \boldsymbol{\tau}, \boldsymbol{\tau}^f_{jH})$, a world competitive equilibrium comprises an aggregate allocation $(\boldsymbol{C}, \boldsymbol{C}^*, \boldsymbol{\ell}, \boldsymbol{\ell}^*, \boldsymbol{y}, \boldsymbol{y}^*)$, prices $(\boldsymbol{w}, \boldsymbol{w}^*, \boldsymbol{P}, \boldsymbol{P}^*, \boldsymbol{p}_{j\omega}, \boldsymbol{R}, \boldsymbol{\mathcal{E}})$, individual policy functions $\{c_t(a,z), c^*_t(a,z)\}$, and joint densities $\{g_t(a,z), g^*_t(a,z)\}$, such that: (i) households, firms and unions optimize, (ii) markets clear, (iii) and the evolution of the joint densities is consistent with household behavior.*

Comparison benchmarks. In our numerical experiments, we contrast optimal tariff policy in this HANK model with two instructive comparison benchmarks. Our model nests a standard representative agent New Keynesian model with trade, which we refer to as the "RANK benchmark". This model features a representative, permanent-income household as in Section 3.1. We also compare our results to the classic tariff formula, where tariffs are not set optimally but instead ad-hoc according to the policy rule (1).

5.2 Calibration

We calibrate the model to match trade flows, trade elasticities, and key moments of the income and wealth distribution at a quarterly frequency. Our targets for trade moments are OECD country averages. Our baseline calibration presently focuses on the case where Home and Foreign each produce a single differentiated trade good. In ongoing work, we extend this to a calibration with many sectors and two countries, the U.S. as Home and an aggregation of the rest of the world as Foreign, using trade and production data from the OECD Inter-Country Input-Output tables. We summarize our calibration targets and parameter values below in Table 1.

Preferences. We set the discount rate $1/\beta - 1$ to a quarterly 0.02 and adopt isoelastic preferences, with $u(c) = \frac{1}{1-\gamma}c^{1-\gamma}$ and $v(\ell) = \frac{1}{1+\nu}\ell^{1+\nu}$. The coefficient of relative risk aversion is calibrated to $\gamma = 2$, and the inverse Frisch elasticity to $\phi = 2.5$ following Chetty et al. (2011).

Labor market structure. We set the elasticity of substitution between labor varieties to $\epsilon^w = 10$ for both Home and Foreign. This is implies moderate wage markups and is consistent with standard estimates from the wage rigidity literature (Auclert et al., 2024b).

Markups. Steady state markups in our model are given by $\frac{\epsilon_j}{\epsilon_j - 1}$. Following standard practice, our baseline calibration sets $\epsilon_j = 10$, reflecting a corporate profit share of around 10%.

Trade elasticities. We set the long-run elasticity of substitution between Home and Foreign varieties to $\eta = \eta^* = 6$ for both economies. Head and Mayer (2014) provide a comprehensive survey of trade elasticity estimates, reporting values typically in a range from 4 to 6. Similarly, Obstfeld and Rogoff (2000) present panel estimates ranging from 5 to 6.

Country size. Following common practice in the New Keynesian literature that studies two-country open economy models, our baseline calibration assumes that Home and Foreign are of equal economic size. We follow Auray et al. (2025) among others.

Table 1. List of Calibrated Parameters in Baseline HANK Model with Trade

	Parameters	Value	Target / Source
ρ	Discount rate (p.q.)	2 %	Standard
γ	Relative risk aversion	2	Standard
ν	Inverse Frisch elasticity	2.5	Chetty et al. (2011)
$\eta = \eta^*$	Trade elasticity of substitution	6	Head and Mayer (2014)
$\theta = \theta^*$	Openess	0.32	Internally calibrated to match OECD trade-to-GDP 63.92%
$A = A^*$	Aggregate productivity	1	Standard

Openess. The home bias parameters θ and θ^* govern the degree of trade openness in our model. We follow Auray et al. (2025) and target trade-to-GDP ratios commonly observed across OECD countries. The trade-to-GDP ratio is computed as a country's imports plus exports divided by GDP. In particular, we calibrate symmetric $\theta = \theta^*$ to match the OECD average trade-to-GDP ratio of 63.93% in 2022. This calibration ensures that our model accurately reflects the openness of a typical developed economy to international trade.

5.3 The Positive Effects of Tariff Shocks

We start our numerical analysis with an exploration of the positive effects of tariff shocks. Figure 5 plots impulse responses to transitory (yellow dashed line) and permanent (blue solid line) import tariff shocks. The permanent shock raises Home import tariffs from 0% to 10% at date 0. The transitory shock also raises tariffs to 10% but gradually reverts them back to 0. In both cases, we initialize the economy at a competitive stationary equilibrium with a 0% tariff. Finally, Figure 5 is plotted under the policy regime in which monetary policy restores the flexible price allocation.

Transitory shock. The import tariff shock raises the price of imported goods. The price of the Home consumption bundle (CPI) rises by 1.5% on impact. In response, Home households reduce their consumption on impact, as the immediate increase in consumer prices lowers real purchasing power. The tariff generates fiscal revenues that are redistributed lump-sum, which partially offsets the decline in real income but not sufficiently to prevent an initial fall in consumption. Following this initial decline, consumption gradually recovers and eventually overshoots its steady state level before returning. This intertemporal substitution pattern is driven partly by the real interest rate, which initially rises, incentivizing households to save. Labor supply rises modestly on impact and its transition dynamics inversely mirror that of consumption. As a result, Home production rises modestly on impact, before falling gradually and then reverting to steady state.

Permanent shock. Under a permanent tariff shock, the economy gradually converges to a new steady state over the course of roughly 40 quarters. Unlike in the representative agent benchmark, this transition is not instantaneous because it takes time for the wealth distribution to converge to the new stationary equilibrium. Directionally, the impulse responses are similar to those under the

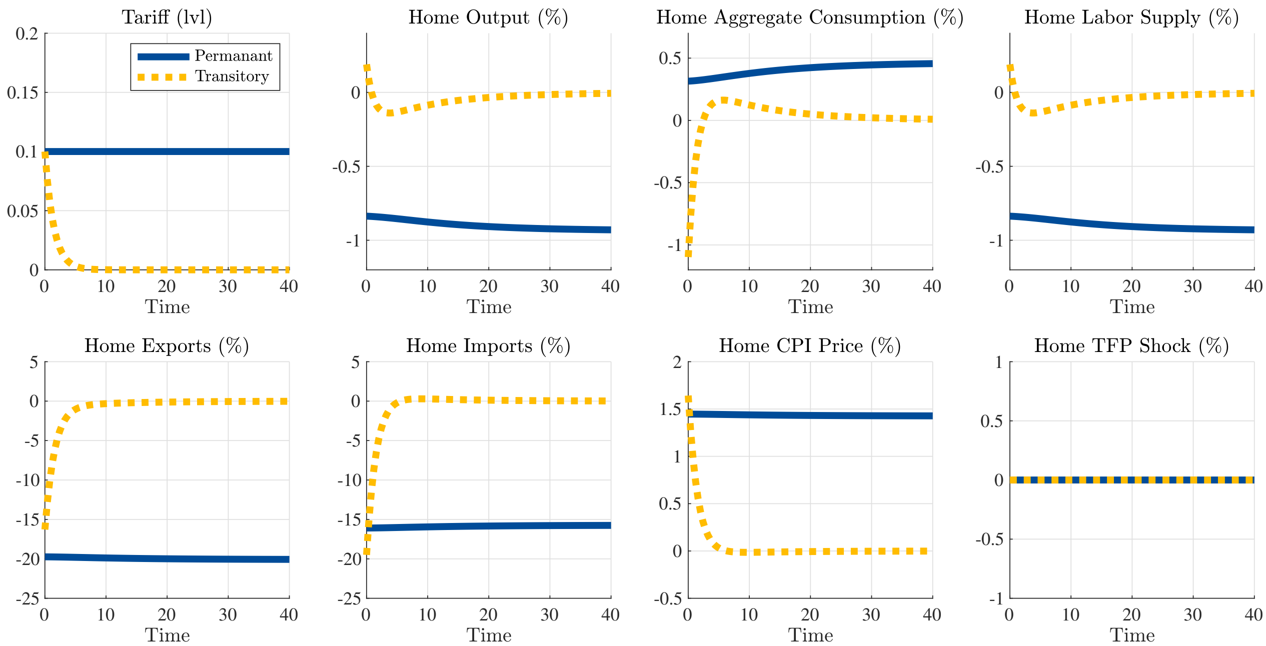

Figure 5. Impulse responses to permanent and transitory tariff shocks

transitory tariff shock. The main difference is that aggregate consumption rises both on impact and in the long run. As a result, labor supply and aggregate production fall both on impact and in the long run.

Effects on the trade balance. In the face of a transitory or permanent import tariff shock, Home imports and exports both fall. The response of Home's trade balance does depend on the shock's persistence, however. Following a transitory tariff shock, imports decline by more than exports on impact, improving Home's trade balance. Following the permanent tariff shock of 10%, however, exports decline by around 20% while imports only fall by around 15%, leading to a deterioration of the long-run trade balance.

5.4 Optimal Tariffs

Ramsey steady state. In Appendix D, we present the Ramsey problem for Home's optimal import tariffs and derive the optimality conditions that characterize the Ramsey plan. In our numerical analysis, the Ramsey plan converges to a stationary equilibrium, which we refer to as the Ramsey steady state (RSS). And the optimal long-run import tariff associated with this RSS is 5.4%.

Optimal tariff dynamics. Finally, we study how the optimal import tariff responds to a TFP shock in Home. We adopt a timeless perspective around the Ramsey steady state, following Dávila and Schaab (2023a). That is, we initialize the economy at the RSS and then compute the optimal tariff

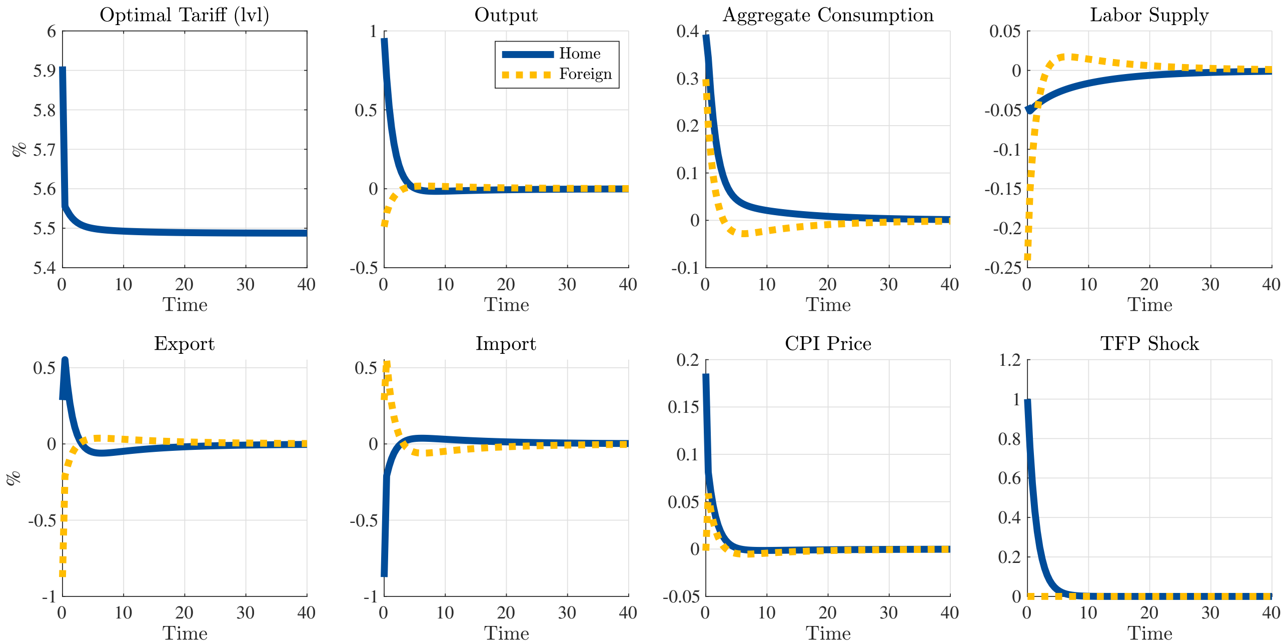

Figure 6. Impulse responses to Home TFP shock under optimal tariffs

transition dynamics as if the planner had solved the Ramsey plan infinitely far in anticipation of the MIT shock. Formally, we solve an augmented Ramsey problem, where we add timeless penalties as defined in Dávila and Schaab (2023a). The key property of this augmented, or timeless, Ramsey problem is that the planner would not want to adjust tariffs away from their Ramsey steady state level in the absence of shocks.

Figure 6 plots the dynamics of timeless optimal import tariffs in response to a 1% positive TFP shock at Home, as well as the impulse responses of key quantities and prices. We continue to assume in this subsection that monetary policy implements the flexible-price allocation in both countries.

The optimal import tariff (top left panel) temporarily rises to around 5.9% before monotonically converging back to its RSS level of 5.4%. A 1% increase in TFP therefore translates into a 0.5% optimal tariff increase. Higher TFP raises production at Home by close to but less than 1%, reflecting a modestly negative labor supply response. Absent policy intervention, the rise in productivity would lower domestic prices and therefore worsen Home's terms of trade. By raising import tariffs, the planner curbs imports and boosts exports for some time. The optimal tariff response therefore delivers a sizeable but short-lived improvement in Home's trade balance. In Foreign, output contracts due to a strong, negative labor supply response. After the initial tariff hike has dissipated, Foreign production rises slightly above Home production for some time.

6 Conclusion

The paper presents an intertemporal tariff formula that generalizes the classic tariff formula to dynamic heterogeneous agent economies. Intertemporal export supply elasticities (iXEs) are sufficient statistics for the optimal tariff, together with tariff revenue weights. We characterize the shape of the iXE matrix $\boldsymbol{\mathcal{E}}$ across several benchmark models of intertemporal trade. When households are permanent-income consumers, the off-diagonal entries of $\boldsymbol{\mathcal{E}}$ are flat within each column. In this case, our intertemporal tariff formula collapses to a static equation, and optimal tariffs are determined exclusively by contemporaneous export supply elasticities. This coincidence breaks down when financial markets are incomplete. In this case, intertemporal export supply elasticities play an important role in the determination of optimal tariffs.

Our intertemporal tariff formula characterizes the import tariff that helps decentralize the Pareto efficient Home allocation. When the first-best allocation is not attainable, the planner has an incentive to use tariffs not solely for terms of trade manipulation but also to tackle other inefficiencies in the economy. We derive a Ramsey targeting rule for optimal second-best tariffs in a large class of heterogeneous agent economies. The second-best tariff trades off intertemporal terms of trade manipulation against gains from production efficiency, risk-sharing, and redistribution.

Finally, we leverage our analytical results to solve for optimal tariff policy in a quantitative heterogeneous agent New Keynesian (HANK) model with trade. The optimal long-run import tariff that emerges in the Ramsey steady state is 5.4%. And from the timeless perspective, the optimal import tariff rises in response to positive TFP shocks.

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A Proofs

A.1 Proof of Lemma 1

Proof. Suppose that an allocation $\{c_t^i(s^t), \ell_t^i(s^t)\}$ and prices $\boldsymbol{p}$ satisfy the implementability condition (10). First, we set $\ell_t = \int_0^1 z_t^i(s^t)\ell_t^i(s^t)\, di$ and $y_t = F_t(\ell_t)$. Then, we set $x_t^* = \mathcal{X}_t^*(\boldsymbol{p})$ as in (9), so that the world resource constraint for goods $x_t^* = c_t - y_t$ is satisfied. The allocation is therefore feasible and it is implementable according to (9).

Conversely, suppose that an allocation $\{c_t^i(s^t), \ell_t^i(s^t)\}$ and prices $\boldsymbol{p}$ are feasible and satisfy (9). Then we can plug in for $y_t = F_t(\ell_t)$ and use the resource constraint for labor, and from the resource constraint for goods we immediately get (10). ■

A.2 Proof of Proposition 1

We first present Proposition 1 again, listing the complete set of efficiency conditions explicitly. Then we present a proof.

Proposition 1 (Home Efficiency)**.** *A Home allocation $\{c_t^i(s^t), \ell_t^i(s^t)\}_{i,t,s^t}$ and prices $\boldsymbol{p}$ are efficient if they satisfy:*

(i) The marginal rate of substitution between consumption and labor is equalized with the marginal rate of transformation for all individuals,

$$-\frac{u^i_{\ell,t}(s^t)}{u^i_{c,t}(s^t)} = F_{\ell,t} z_t^i(s^t).$$

(ii) The marginal rates of substitution between consumption across different histories are equalized across individuals,

$$\frac{u^i_{c,t}(s^t)}{u^i_{c,k}(s^k)} = \frac{u^j_{c,t}(s^t)}{u^j_{c,k}(s^k)}.$$

(iii) Probability-weighted marginal utilities are equalized across all idiosyncratic histories s^t and $\tilde{s}^t$ at date t,

$$\pi(s^t)u^i_{c,t}(s^t) = \pi(\tilde{s}^t)u^i_{c,t}(\tilde{s}^t).$$

(iv) The intertemporal terms of trade efficiency condition is satisfied

$$0 = \sum_k MRS_{kt} \frac{\partial \mathcal{X}_k^*}{\partial p_t}$$

where MRS_{kt} is the marginal rate of substitution between consumption at dates k and t, which is equalized across individuals and across histories at k and t.

Proof. We present the Lagrangian again, which is

$$L = \int_0^1 \alpha^i \sum_t \beta^t \sum_{s^t} \pi(s^t) u^i(c_t^i(s^t), \ell_t^i(s^t))\, di + \sum_t \beta^t \mu_t \left[\mathcal{X}_t^*(\boldsymbol{p}) - \int_0^1 c_t^i(s^t) di + F_t\left(\int_0^1 z_t^i(s^t) \ell_t^i(s^t) di \right) \right].$$

The first-order condition for consumption $c_t^i(s^t)$ is given by

$$\alpha^i \pi(s^t) u_{c,t}^i(s^t) = \mu_t.$$

The first-order condition for labor supply $\ell_t^i(s^t)$ is given by

$$\alpha^i \pi(s^t) u_{\ell,t}^i(s^t) = -F_{\ell,t} \mu_t z_t^i(s^t).$$

And the first-order condition for the price p_t is given by

$$0 = \sum_k \beta^k \mu_k \frac{\partial \mathcal{X}_k^*}{\partial p_t}.$$

We now rearrange these equations. First, we get the MRS = MRT efficiency condition by solving the first two FOCs for μ_t, yielding

$$u_{\ell,t}^i(s^t) = -F_{\ell,t} u_{c,t}^i(s^t) z_t^i(s^t).$$

where the α^i drop out because we are comparing MRS and MRT for a given individual i.

Second, we get the MRS equalization condition by solving out for μ_t in the first FOC across two individuals i and j, and across two different histories s^t and s^k. We can write

$$\alpha^i u_{c,t}^i(s^t) = \alpha^j u_{c,t}^j(s^t)$$

We now divide this equation for date t and history s^t by the same equation for date k and history s^k. The Pareto weights α^i and α^j drop out, and we arrive at the efficiency condition of the Proposition.

Third, we get one additional efficiency condition because we focus on idiosyncratic risk, which implies that the Lagrange multiplier μ_t is not contingent on the history. Therefore, we can again use the first FOC and compare across two different histories s^t and $\tilde{s}^t$ at the same date within individual i, yielding

$$\pi(s^t) u_{c,t}^i(s^t) = \pi(\tilde{s}^t) u_{c,t}^i(\tilde{s}^t),$$

which is the desired condition.

Finally, notice that all MRS are equalized across individuals, which implies that we can define the economy-wide MRS for Home as

$$MRS_{kt} = \frac{\beta^k \pi(s^k) u^i_{c,k}(s^k)}{\beta^t \pi(s^t) u^i_{c,t}(s^t)}$$

which is equal across all i. Using the first FOC to solve out for μ_t, we can therefore rewrite the third FOC for the price as

$$0 = \sum_k \beta^k \alpha^i \pi(s^k) u^i_{c,k}(s^k) \frac{\partial \mathcal{X}^*_k}{\partial p_t} = \sum_k \frac{\beta^k \pi(s^k) u^i_{c,k}(s^k)}{\beta^t \pi(s^t) u^i_{c,t}(s^t)} \frac{\partial \mathcal{X}^*_k}{\partial p_t},$$

which concludes the proof. ■

A.3 Proof of Proposition 2

Proof. The proof is presented in the main text. ■

A.4 Proof of Proposition 3

Proof. The Ramsey problem for optimal second-best tariffs in the environment of Section 2.5 is given by

$$\max \int_0^1 \alpha^i \sum_t \beta^t \sum_{s^t} \pi(s^t) u^i(c^i_t(s^t), \ell^i_t(s^t))\, di,$$

subject to

$$c^i_t(s^t) = \mathcal{C}^i_t(\boldsymbol{p}, \boldsymbol{\tau}, s^t)$$

$$\ell^i_t(s^t) = \mathcal{L}^i_t(\boldsymbol{p}, \boldsymbol{\tau}, s^t)$$

$$\mathcal{X}^*_t(\boldsymbol{p}) = \int_0^1 c^i_t(s^t)\, di - F_t\left(\int_0^1 z^i_t(s^t) \ell^i_t(s^t)\, di \right)$$

We refer to the first and second constraint as internal implementability conditions and to the third constraint as the external implementability condition.

Our proof strategy is as follows, building on Dávila and Schaab (2024) and Dávila and Schaab (2023b): We consider a feasible and implementable perturbation $d\tau_k$. When evaluated at the Ramsey plan, the planner must be indifferent to such a perturbation from the perspective of date 0. That is, we must have

$$0 = \frac{dW_0}{d\tau_k} = \sum_i \frac{\partial \mathcal{W}}{\partial V^i_0} \frac{dV^i_0}{d\tau_k}$$

for all k around the Ramsey plan. Here, $\alpha^i = \frac{\partial \mathcal{W}}{\partial V^i_0}$ denotes the marginal contribution of individual i to social welfare under the social welfare function $\mathcal{W}$. We proceed in 5 steps.

Step 1 (Welfare Numeraire). We express the welfare assessment in a comparable unit which we call the welfare numeraire. For simplicity, we express interpersonal comparisons in units of consumption at date 0. Therefore, the Ramsey FOC can be written as

$$0 = \frac{1}{\int_0^1 \alpha^i \lambda^i di} \frac{dW_0}{d\tau_k} = \frac{1}{\int_0^1 \alpha^i \lambda^i di} \int_0^1 \alpha^i \lambda^i \frac{1}{\lambda^i} \frac{dV_0^i}{d\tau_k} di$$

where $\lambda^i = u^i_{c,0}$ is the normalizing factor that translates individual gains and losses into units of date 0 consumption. Multiplying and dividing by λ^i accomplishes that $\frac{1}{\lambda^i}\frac{dV_0^i}{d\tau_k}$ now captures individual i's willingness to pay for the tariff perturbation $d\tau_k$ in (consumption-equivalent) units of date 0 consumption. We can therefore rewrite the Ramsey FOCs as

$$0 = \int_0^1 \omega^i \frac{1}{\lambda^i} \frac{dV_0^i}{d\tau_k} di,$$

where $\omega^i = \frac{\alpha^i \lambda^i}{\int_0^1 \alpha^i \lambda^i di}$ is the normalized individual welfare weight expressed in units of welfare numeraire.

Step 2 (Redistribution). We first show that optimal tariff policy trades off efficiency and redistribution considerations. Noticing that $\int_0^1 \omega^i di = 1$, a covariance decomposition of the Ramsey FOCs yields

$$0 = \int_0^1 \frac{1}{\lambda^i} \frac{dV_0^i}{d\tau_k} di + \mathbb{C}ov_i\left(\omega^i, \frac{1}{\lambda^i}\frac{dV_0^i}{d\tau_k}\right),$$

where the first term captures all efficiency gains from tariff perturbation $d\tau_k$ and the second term captures redistribution gains (Dávila and Schaab, 2024).

In the main text, we assume that individuals are ex ante homogeneous, in which case $\omega^i = \omega^j$ for all i and j, and so the redistribution term drops out. We maintain this assumption in the remainder of the proof. A general targeting formula for optimal second-best tariffs will include the redistribution motive $\mathbb{C}ov_i(\omega^i, \frac{1}{\lambda^i}\frac{dV_0^i}{d\tau_k})$.

Step 3 (Risk-Sharing). We now start unpacking the efficiency gains from optimal tariff policy. Notice that efficiency can be written as

$$\int_0^1 \frac{1}{\lambda^i} \sum_t \beta^t \sum_{s^t} \pi(s^t) \left[u^i_{c,t}(s^t) \frac{dc^i_t(s^t)}{d\tau_k} + u^i_{\ell,t}(s^t) \frac{d\ell^i_t(s^t)}{d\tau_k} \right] di$$

after using the definition of i's lifetime utility. Notice that here and throughout it is understood that the total derivatives refer to

$$\frac{dc^i_t(s^t)}{d\tau_k} = \frac{\partial \mathcal{C}^i_t(\boldsymbol{r}, \boldsymbol{\tau}, s^t)}{\partial \tau_k} + \sum_s \frac{\partial \mathcal{C}^i_t(\boldsymbol{r}, \boldsymbol{\tau}, s^t)}{\partial r_s} \frac{dr_s}{d\tau_k}$$

and similarly for $\frac{d\ell_t^i(s^t)}{d\tau_k}$, where $\frac{dr_s}{d\tau_k}$ is determined by the implementability condition (10).

We also assume in the main text that, conditional on the aggregate labor wedge, individuals are then on their individual labor-leisure conditions. That is,

$$u_{\ell,t}^i(s^t) = -z_t^i(s^t)F_{\ell,t}(1-\Lambda_t^\ell)u_{c,t}^i(s^t),$$

which allows us to rewrite the efficiency term simply as

$$0 = \int_0^1 \frac{1}{\lambda^i}\sum_t \beta^t \sum_{s^t} \pi(s^t)u_{c,t}^i(s^t)\left[\frac{dc_t^i(s^t)}{d\tau_k} - z_t^i(s^t)F_{\ell,t}(1-\Lambda_t^\ell)\frac{d\ell_t^i(s^t)}{d\tau_k}\right] di.$$

Going forward, we denote by

$$\frac{dV_t^i(s^t)}{d\tau_k} = \frac{dc_t^i(s^t)}{d\tau_k} - z_t^i(s^t)F_{\ell,t}(1-\Lambda_t^\ell)\frac{d\ell_t^i(s^t)}{d\tau_k}$$

the consumption-equivalent welfare gain of individual i at date t in history s^t from the tariff perturbation $d\tau_k$.

We therefore have

$$\begin{aligned} 0 &= \int_0^1 \frac{1}{\lambda^i}\sum_t \beta^t \sum_{s^t} \pi(s^t)u_{c,t}^i(s^t)\frac{dV_t^i(s^t)}{d\tau_k}\, di \\ &= \int_0^1 \sum_t \sum_{s^t} \frac{\beta^t \pi(s^t)u_{c,t}^i(s^t)}{u_{c,0}^i}\frac{dV_t^i(s^t)}{d\tau_k}\, di \\ &= \int_0^1 \sum_t \sum_{s^t} \frac{\beta^t \sum_{s^t}\pi(s^t)u_{c,t}^i(s^t)}{u_{c,0}^i}\frac{\pi(s^t)u_{c,t}^i(s^t)}{\sum_{s^t}\pi(s^t)u_{c,t}^i(s^t)}\frac{dV_t^i(s^t)}{d\tau_k}\, di \end{aligned}$$

where we plugged in for $\lambda^i = u_{c,0}^i$. We have expressed the efficiency gain of the tariff perturbation in terms of three key objects: First, the consumption-equivalent welfare gain $\frac{dV_t^i(s^t)}{d\tau_k}$ records individual i's willingness to pay for the allocation change the perturbation induces at date t in history s^t. Second, $\frac{\pi(s^t)u_{c,t}^i(s^t)}{\sum_{s^t}\pi(s^t)u_{c,t}^i(s^t)}$ captures individual i's marginal rate of substitution between a unit of consumption in history s^t and a unit of consumption in all histories at date t. Dispersion in this MRS across histories indicates that individual i is not able to smooth marginal utilities across histories. Third, $\frac{\beta^t\sum_{s^t}\pi(s^t)u_{c,t}^i(s^t)}{u_{c,0}^i}$ captures the MRS of individual i between a unit of consumption at date t (in all histories s^t) and a unit of consumption at date 0.

Under the assumption of ex ante homogeneous households, which we make in the main text, we have

$$\frac{\beta^t \sum_{s^t}\pi(s^t)u_{c,t}^i(s^t)}{u_{c,0}^i} = \frac{\beta^t \sum_{s^t}\pi(s^t)u_{c,t}^j(s^t)}{u_{c,0}^j} = \omega_t$$

for all i and j. In other words, all individuals have the same valuation of date t consumption in expectation. This would not be the case if individuals were already heterogeneous at date 0. But under this simplifying assumption, we can now rewrite the Ramsey FOC as

$$0 = \sum_t \omega_t \sum_{s^t} \int_0^1 \frac{\pi(s^t) u^i_{c,t}(s^t)}{\sum_{s^t} \pi(s^t) u^i_{c,t}(s^t)} \frac{dV^i_t(s^t)}{d\tau_k} \, di.$$

One final cross-sectional covariance decomposition then yields

$$0 = \underbrace{\sum_t \omega_t \sum_{s^t} \int_0^1 \frac{dV^i_t(s^t)}{d\tau_k} \, di}_{\text{Aggregate Efficiency}} + \underbrace{\sum_t \omega_t \sum_{s^t} \mathbb{C}ov_i \left(\frac{\pi(s^t) u^i_{c,t}(s^t)}{\sum_{s^t} \pi(s^t) u^i_{c,t}(s^t)}, \; \frac{dV^i_t(s^t)}{d\tau_k} \right)}_{\text{Risk-Sharing}}$$

since

$$\int_0^1 \frac{\pi(s^t) u^i_{c,t}(s^t)}{\sum_{s^t} \pi(s^t) u^i_{c,t}(s^t)} \, di = 1$$

when individuals are ex ante homogeneous due to a law of large numbers. We have therefore decomposed the efficiency gain of the tariff perturbation into an aggregate efficiency component and a risk-sharing component.

Step 4 (Intertemporal Terms of Trade Manipulation) Finally, we now unpack the aggregate efficiency gains from the tariff perturbation into a production efficiency gain and a terms of trade manipulation gain. Aggregate efficiency is given by

$$\begin{aligned}
\sum_t \omega_t \sum_{s^t} \int_0^1 \frac{dV^i_t(s^t)}{d\tau_k} \, di &= \sum_t \omega_t \sum_{s^t} \int_0^1 \left[\frac{dc^i_t(s^t)}{d\tau_k} - z^i_t(s^t) F_{\ell,t} (1 - \Lambda^\ell_t) \frac{d\ell^i_t(s^t)}{d\tau_k} \right] di \\
&= \sum_t \omega_t \frac{dc_t}{d\tau_k} - \sum_t \omega_t F_{\ell,t} (1 - \Lambda^\ell_t) \sum_{s^t} \int_0^1 z^i_t(s^t) \frac{d\ell^i_t(s^t)}{d\tau_k} \, di \\
&= \sum_t \omega_t \frac{dc_t}{d\tau_k} - \sum_t \omega_t F_{\ell,t} (1 - \Lambda^\ell_t) \frac{d\ell_t}{d\tau_k}
\end{aligned}$$

Using the external implementability condition (10), we can rewrite this as

$$\begin{aligned}
\sum_t \omega_t \left[\frac{dc_t}{d\tau_k} - F_{\ell,t} (1 - \Lambda^\ell_t) \frac{d\ell_t}{d\tau_k} \right] &= \sum_t \omega_t \left[\frac{dc_t}{d\tau_k} - F_{\ell,t} \frac{d\ell_t}{d\tau_k} \right] + \sum_t \omega_t F_{\ell,t} \Lambda^\ell_t \frac{d\ell_t}{d\tau_k} \\
&= \sum_t \omega_t \frac{dx^*_t}{d\tau_k} + \sum_t \omega_t \Lambda^\ell_t \frac{dy_t}{d\tau_k}.
\end{aligned}$$

And finally, we use the sequence-space representation of Foreign $x^*_t = \mathcal{X}^*_t(\boldsymbol{p})$ to write

$$\sum_t \omega_t \sum_s \frac{\partial \mathcal{X}^*_t}{\partial p_s} \frac{dp_s}{d\tau_k} + \sum_t \omega_t \Lambda^\ell_t \frac{dy_t}{d\tau_k}.$$

Notice that we can rewrite the first term as

$$\sum_t \omega_t \sum_s \frac{\partial \mathcal{X}_t^*}{\partial p_s} \frac{dp_s}{d\tau_k} = \sum_s \sum_t \omega_t \frac{\partial \mathcal{X}_t^*}{\partial p_s} \frac{dp_s}{d\tau_k} = \sum_s \Lambda_s^{\text{ITM}} \frac{dp_s}{d\tau_k},$$

where $\Lambda_s^{\text{ITM}} = \sum_t \omega_t \frac{\partial \mathcal{X}_t^*}{\partial p_s}$, which concludes the proof.

■

A.5 Proof of Lemma 2

Proof. Most of the proof is presented constructively in the main text. To fill in the missing steps, notice that

$$\begin{aligned}
\frac{\partial \log \mathcal{C}_k^*}{\partial \log p_t} &= \frac{p_t}{c_k^*} \frac{\partial \mathcal{C}_k^*}{\partial p_t} \\
&= \frac{p_t}{c_k^*} \frac{p_k}{\beta^k u_{cc,k}^*} \frac{\partial \lambda^*}{\partial p_t} + \frac{p_t}{c_k^*} \frac{\lambda^*}{\beta^k u_{cc,k}^*} \mathbb{1}_{k=t} \\
&= \frac{p_t}{c_k^*} \frac{p_k}{\beta^k u_{cc,k}^*} \frac{\lambda^*}{\lambda^*} \frac{u_{c,k}^*}{u_{c,k}^*} \frac{\partial \lambda^*}{\partial p_t} + \frac{p_t}{c_k^*} \frac{p_k}{p_k} \frac{u_{c,k}^*}{u_{c,k}^*} \frac{\lambda^*}{\beta^k u_{cc,k}^*} \mathbb{1}_{k=t} \\
&= \frac{1}{c_k^*} \frac{u_{c,k}^*}{u_{cc,k}^*} \frac{\partial \log \lambda^*}{\partial \log p_t} + \frac{p_t}{c_k^*} \frac{1}{p_k} \frac{u_{c,k}^*}{u_{cc,k}^*} \mathbb{1}_{k=t} \\
&= \frac{1}{c_k^*} \frac{u_{c,k}^*}{u_{cc,k}^*} \frac{\partial \log \lambda^*}{\partial \log p_t} + \frac{1}{c_t^*} \frac{u_{c,t}^*}{u_{cc,t}^*} \mathbb{1}_{k=t}
\end{aligned}$$

where we last line follows from noting that the second term only survives when $k = t$, which allows us to change the time subscripts. Finally, we define the inverse IES as $\gamma_t^* = -\frac{c_t^* u_{cc,t}^*}{u_{c,t}^*}$, which leads to

$$\frac{\partial \log \mathcal{C}_k^*}{\partial \log p_t} = -\frac{1}{\gamma_k^*} \frac{\partial \log \lambda^*}{\partial \log p_t} - \frac{1}{\gamma_k^*} \mathbb{1}_{k=t}$$

as in the Lemma.

Next, we denote by $\boldsymbol{\rho}$ the vector whose t-th entry is $\frac{\partial \log \lambda^*}{\partial \log p_t}$, and by $\boldsymbol{\phi}$ the vector whose t-th entry is $-\frac{1}{\gamma_t^*}$. Then we have

$$\frac{\partial \log \mathcal{C}_k^*}{\partial \log p_t} = \boldsymbol{\phi}_k \mathbb{1}_{k=t} + \boldsymbol{\phi}_k \boldsymbol{\rho}_t$$

or in matrix notation

$$\boldsymbol{\mathcal{C}} = \text{diag}(\boldsymbol{\phi}) + \boldsymbol{\phi} \boldsymbol{\rho}'.$$

Therefore, the log sequence-space Jacobian $\boldsymbol{\mathcal{C}}$ is the sum of a diagonal matrix and an outer product of two vectors. We denote the former by $\boldsymbol{D} = \text{diag}(\boldsymbol{\phi})$ and the latter by $\boldsymbol{P} = \boldsymbol{\phi}\boldsymbol{\rho}'$. A matrix formed by the outer product of two vectors has rank one.

Since we are dealing with infinite vectors and matrices, we work on the space of bounded sequences

$$\ell^\infty = \left\{ \boldsymbol{z} = \{z_t\}_{t\geq 0} : ||z||_\infty = \sup_{t\geq 0} \boldsymbol{z} < \infty \right\}$$

We have $\boldsymbol{\phi} \in \ell^\infty$ and $\boldsymbol{\rho} \in \ell^\infty$. This implies that their outer product $\boldsymbol{P}$ is bounded on ℓ^∞ and of rank one. Also, notice that $\kappa = \sum_t \rho_t$ exists.

Finally, consider the case with CRRA consumption preferences and $\gamma_t^* = \gamma$. In that case, we have

$$\boldsymbol{\mathcal{C}} = -\frac{1}{\gamma}\boldsymbol{I} - \frac{1}{\gamma}\boldsymbol{1}\boldsymbol{\rho}',$$

where $\boldsymbol{I}$ is the identity matrix and $\boldsymbol{1}$ is the infinite column vector of 1s. This implies that $-\frac{1}{\gamma}$ is one eigenvalue of infinite multiplicity. There can only be at most one additional eigenvalue since $\boldsymbol{\mathcal{C}}$ in this case represents a rank-one shift of a multiple of the identity matrix. This eigenvalue is given by $-\frac{1}{\gamma}(1+\kappa)$.

■

A.6 Proof of Proposition 4

Proof. We now prove that the intertemporal tariff formula collapses to the static one presented in the main text. First, notice that we can write

$$\begin{aligned}
\frac{1}{\tau_t} &= \sum_k \omega_{kt} \frac{\partial \log \mathcal{X}_k^*}{\partial \log p_t} \\
&= \sum_k \omega_{kt} \frac{p_t}{x_k^*} \frac{\partial \mathcal{X}_k^*}{\partial p_t} \\
&= -\sum_k \omega_{kt} \frac{p_t}{x_k^*} \frac{\partial \mathcal{C}_k^*}{\partial p_t} \\
&= -\sum_k \omega_{kt} \frac{p_t}{x_k^*} \frac{\partial \tilde{\mathcal{C}}_k^*}{\partial \lambda^*} \frac{\partial \lambda^*}{\partial p_t} - \frac{p_t}{x_t^*} \frac{\partial \tilde{\mathcal{C}}_t^*}{\partial p_t}
\end{aligned}$$

All we have to show is that the first term is 0, because from there it immediately follows that

$$\frac{1}{\tau_t} = -\frac{p_t}{x_t^*} \frac{\lambda^*}{\beta^t u_{cc,t}^*} = -\frac{1}{x_t^*} \frac{\lambda^* p_t}{\beta^t u_{c,t}^*} \frac{u_{c,t}^*}{u_{cc,t}^*} = -\frac{1}{x_t^*} \frac{u_{c,t}^*}{u_{cc,t}^*}$$

as in the main text.

To show that the first term is 0 for all t. We have

$$0 = -\sum_k \omega_{kt} \frac{p_t}{x_k^*} \frac{\partial \tilde{\mathcal{C}}_k^*}{\partial \lambda^*} \frac{\partial \lambda^*}{\partial p_t} \qquad \Longleftrightarrow \qquad 0 = \sum_k \omega_{kt} \frac{p_t}{x_k^*} \frac{\partial \tilde{\mathcal{C}}_k^*}{\partial \lambda^*} = \sum_k \frac{\tau_k p_k x_k^*}{\tau_t p_t x_t^*} \frac{p_t}{x_k^*} \frac{\partial \tilde{\mathcal{C}}_k^*}{\partial \lambda^*}$$

And this will be true if and only if

$$0 = \sum_k \tau_k p_k \frac{\partial \tilde{\mathcal{C}}_k^*}{\partial \lambda^*} = \sum_k \tau_k p_k \frac{p_k}{\beta^k u''(c_k^*)} \qquad \Longleftrightarrow \qquad 0 = \sum_k \tau_k p_k \frac{u'(c_k^*)}{u''(c_k^*)},$$

where we used $\beta^k u'(c_k^*) = \lambda^* p_k$ and the fact that λ^* comes out of the sum.

Next, define

$$Z = \sum_k \tau_k p_k \frac{u'(c_k^*)}{u''(c_k^*)}$$

Then we can write the intertemporal tariff formula as

$$\begin{aligned}
\frac{1}{\tau_t} &= -\frac{\partial \lambda^*}{\partial p_t} \frac{1}{\tau_t x_t^*} \sum_k \tau_k p_k \frac{\partial \tilde{\mathcal{C}}_k^*}{\partial \lambda^*} - \frac{p_t}{x_t^*} \frac{\partial \tilde{\mathcal{C}}_t^*}{\partial p_t} \\
&= -\frac{\partial \lambda^*}{\partial p_t} \frac{1}{\tau_t x_t^*} \frac{1}{\lambda^*} Z - \frac{p_t}{x_t^*} \frac{\partial \tilde{\mathcal{C}}_t^*}{\partial p_t} \\
&= -\frac{\partial \log \lambda^*}{\partial \log p_t} \frac{1}{\tau_t p_t x_t^*} Z - \frac{1}{x_t^*} \frac{\lambda^* p_t}{\beta^t u'(c_t^*)} \frac{u'(c_t^*)}{u''(c_t^*)} \\
&= -\frac{\partial \log \lambda^*}{\partial \log p_t} \frac{1}{\tau_t p_t x_t^*} Z - \frac{1}{x_t^*} \frac{u'(c_t^*)}{u''(c_t^*)}
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
p_t x_t^* &= -\frac{\partial \log \lambda^*}{\partial \log p_t} Z - \tau_t p_t \frac{u'(c_t^*)}{u''(c_t^*)} \\
&= -\frac{\partial \log \lambda^*}{\partial \log p_t} Z + \frac{1}{\gamma_t^*} \tau_t p_t c_t^*
\end{aligned}$$

or simply

$$\tau_t = \gamma_t^* \frac{1}{p_t c_t^*} \left(p_t x_t^* + \frac{\partial \log \lambda^*}{\partial \log p_t} Z \right)$$

We can now plug this into the equation for Z, which yields

$$\begin{aligned} Z &= -\sum_k \frac{1}{\gamma_k^*}\tau_k p_k c_k^* \\ &= -\sum_k \left(p_k x_k^* + \frac{\partial \log \lambda^*}{\partial \log p_k} Z \right) \\ &= -\sum_k p_k x_k^* - Z \sum_k \frac{\partial \log \lambda^*}{\partial \log p_k} \end{aligned}$$

or simply

$$Z\left(1 + \sum_k \frac{\partial \log \lambda^*}{\partial \log p_k}\right) = \sum_k p_k x_k^* = -NFA_0^*,$$

which is 0 when $NFA_0^* = 0$ as we assume in Section 3.1.

We can also make the argument in a different way. From the Home efficiency condition with tariff

$$0 = \sum_k (1+\tau_k) p_k \frac{\partial \mathcal{X}_k^*}{\partial p_t}.$$

Also from the lifetime budget constraint, we have

$$0 = x_t^* + \sum_k p_k \frac{\partial \mathcal{X}_k^*}{\partial p_t} \quad \implies \quad x_t^* = \sum_k p_k \frac{\partial \mathcal{C}_k^*}{\partial p_t}$$

Therefore, we have

$$\begin{aligned} -x_t^* &= \sum_k \tau_k p_k \frac{\partial \mathcal{C}_k^*}{\partial p_t} \\ &= \sum_k \tau_k p_k \left[\frac{p_k}{\beta^k u''(c_k^*)} \frac{\partial \lambda^*}{\partial p_t} + \frac{\lambda^*}{\beta^k u''(c_k^*)} \mathbb{1}_{k=t} \right] \\ &= \sum_k \tau_k p_k \frac{p_k}{\beta^k u''(c_k^*)} \frac{\partial \lambda^*}{\partial p_t} + \tau_t p_t \frac{\lambda^*}{\beta^t u''(c_t^*)} \end{aligned}$$

And notice that this can be written as

$$\begin{aligned} -x_t^* &= \frac{\partial \lambda^*}{\partial p_t} \frac{1}{\lambda^*} \sum_k \tau_k p_k \frac{u'(c_k^*)}{u''(c_k^*)} + \tau_t p_t \frac{\lambda^*}{\beta^t u''(c_t^*)} \\ &= \frac{\partial \lambda^*}{\partial p_t} \frac{1}{\lambda^*} Z + \tau_t p_t \frac{\lambda^*}{\beta^t u''(c_t^*)} \end{aligned}$$

Separately, notice that fully differentiating the lifetime budget constraint and the FOC yields

$$x_t^* = \sum_k p_k \frac{\partial c_k^*}{\partial p_t}$$

$$\beta^k u''(c_k^*) \frac{\partial c_k^*}{\partial p_t} = p_k \frac{\partial \lambda^*}{\partial p_t} + \lambda^* \frac{\partial p_k}{\partial p_t}$$

which lets us solve for

$$x_t^* = \sum_k p_k \left[\frac{p_k}{\beta^k u''(c_k^*)} \frac{\partial \lambda^*}{\partial p_t} + \frac{\lambda^*}{\beta^k u''(c_k^*)} \frac{\partial p_k}{\partial p_t} \right]$$

$$x_t^* = \frac{\partial \lambda^*}{\partial p_t} \sum_k \frac{p_k^2}{\beta^k u''(c_k^*)} + p_t \frac{\lambda^*}{\beta^t u''(c_t^*)}$$

or simply

$$\frac{\partial \lambda^*}{\partial p_t} = \frac{1}{\sum_k \frac{p_k^2}{\beta^k u''(c_k^*)}} \left(x_t^* - p_t \frac{\lambda^*}{\beta^t u''(c_t^*)} \right)$$

Now we can put these two together, which yields

$$\begin{aligned} -x_t^* &= \frac{1}{\sum_k \frac{p_k^2}{\beta^k u''(c_k^*)}} \left(x_t^* - p_t \frac{\lambda^*}{\beta^t u''(c_t^*)} \right) \frac{1}{\lambda^*} Z + \tau_t p_t \frac{\lambda^*}{\beta^t u''(c_t^*)} \\ &= \frac{1}{\frac{1}{\lambda^*} \sum_k p_k \frac{u'(c_k^*)}{u''(c_k^*)}} \left(x_t^* - \frac{u'(c_t^*)}{u''(c_t^*)} \right) \frac{1}{\lambda^*} Z + \tau_t \frac{u'(c_t^*)}{u''(c_t^*)} \\ &= \frac{1}{\sum_k p_k \frac{u'(c_k^*)}{u''(c_k^*)}} \left(x_t^* - \frac{u'(c_t^*)}{u''(c_t^*)} \right) Z + \tau_t \frac{u'(c_t^*)}{u''(c_t^*)} \end{aligned}$$

This allows us to solve for the tariff as

$$\tau_t \frac{u'(c_t^*)}{u''(c_t^*)} = -x_t^* - \frac{1}{\sum_k p_k \frac{u'(c_k^*)}{u''(c_k^*)}} \left(x_t^* - \frac{u'(c_t^*)}{u''(c_t^*)} \right) Z$$

or simply

$$\tau_t = -\frac{u''(c_t^*)}{u'(c_t^*)} x_t^* - \frac{1}{\sum_k p_k \frac{u'(c_k^*)}{u''(c_k^*)}} \left(\frac{u''(c_t^*)}{u'(c_t^*)} x_t^* - 1 \right) Z$$

Plugging back into the definition of Z completes the proof.

■

A.7 Proof of Proposition 5

Proof. The proof is presented in the main text. ∎

B A Neoclassical Representative Agent Production Economy

This Appendix presents a variant of the representative agent (RA) model of Section 3.1 with production. We show that the intertemporal export supply function $\mathcal{X}^*$ and its sequence-space Jacobian $\boldsymbol{\mathcal{E}}$ continue to take the special form discussed in Lemma 2 in this production economy.

B.1 Environment

Households. Home and Foreign are each populated by a representative household whose lifetime utilities are

$$V_0 = \sum_t \beta^t u(c_t, \ell_t) \qquad \text{and} \qquad V_0^* = \sum_t \beta^t u(c_t^*, \ell_t^*). \tag{46}$$

We denote the intertemporal price of the single consumption good by p_t. Assuming complete financial markets allows us to write the consumption-savings problems of both households in terms of the lifetime budget constraints

$$0 = \sum_t ((1+\tau_t)p_t c_t - w_t \ell_t + T_t) \qquad \text{and} \qquad 0 = \sum_t (p_t c_t^* - w_t^* \ell_t^*), \tag{47}$$

where w_t and w_t^* denote the local wage rates in both countries, τ_t is Home's import tariff, and T_t is a lump-sum rebate. We assume here for simplicity that both countries' initial net foreign asset positions are 0 and that Foreign does not set a tariff.

Households therfore maximize preferences (46) subject to (47). Denoting the Lagrange multipliers on each lifetime budget constraint by λ and λ^*, the associated first-order conditions are

$$\beta^t u_{c,t} = \lambda(1+\tau_t)p_t \qquad \text{and} \qquad -\frac{u_{\ell,t}}{u_{c,t}} = \frac{w_t}{(1+\tau_t)p_t} \tag{48}$$

for Home and

$$\beta^t u_{c,t}^* = \lambda^* p_t \qquad \text{and} \qquad -\frac{u_{\ell,t}^*}{u_{c,t}^*} = \frac{w_t^*}{p_t} \tag{49}$$

for Foreign, where we use the shorthand notation $u_{c,t} = \frac{\partial}{\partial c_t} u(c_t, \ell_t)$ and so forth.

Firms. A representative and perfectly competitive firm operates a production technology in each country that is linear in labor and given by

$$y_t = A_t \ell_t \qquad \text{and} \qquad y_t^* = A_t^* \ell_t^*, \tag{50}$$

where productivities A_t and A_t^* follow exogenous deterministic paths. We assume that firms at Home are subject to a subsidy on revenue also given by τ_t. Profit maximization therefore implies the optimality conditions

$$w_t = (1+\tau_t)p_t A_t \qquad \text{and} \qquad w_t^* = p_t A_t^*. \tag{51}$$

Government, market clearing, and competitive equilibrium. The Home government runs a balanced budget. This requires that tax receipts are equal to rebates and subsidies,

$$\tau_t p_t c_t = T_t + \tau_t p_t y_t \tag{52}$$

in all periods t. In other words, the government taxes imports when $c_t - y_t > 0$ and subsidizes exports when $c_t - y_t < 0$.

The market clearing condition for the single consumption good at date t is given by

$$y_t + y_t^* = c_t + c_t^*. \tag{53}$$

Market clearing for labor in each country is already implicit in our notation since we do not distinguish between labor demand and supply.

Definition 4 (Competitive Equilibrium)**.** *Taking as given sequences $\boldsymbol{A}$ and $\boldsymbol{A}^*$ as well as Home tariffs $\boldsymbol{\tau}$, a world competitive equilibrium comprises allocations $(\boldsymbol{c}, \boldsymbol{c}^*, \boldsymbol{\ell}, \boldsymbol{\ell}^*, \boldsymbol{y}, \boldsymbol{y}^*, \boldsymbol{T})$, multipliers (λ, λ^*) and prices $(\boldsymbol{p}, \boldsymbol{w}, \boldsymbol{w}^*)$ that satisfy production technologies (50), lifetime budget constraints (47), household optimization in Home (48) and Foreign (49), firm optimization (51), the Home government budget constraint (52), and market clearing (53).*

B.2 Intertemporal Export Supply Elasticities

In the absence of distortions, the Home efficiency condition (12) and optimal tariff formula (13) apply as-is to this production economy. We start with a constructive derivation of the intertemporal export supply function $\mathcal{X}^*$.

The Foreign household problem can then be characterized by the lifetime budget constraint (47) and the two first-order conditions in (49). We use firm optimality condition (51) to solve for the Foreign wage w_t^*. The labor-leisure condition then solves for labor supply ℓ_t^* as a function of consumption c_t^* and exogenous technology A_t^*. The Euler equation for consumption solves for c_t^* as a function of the contemporaneous price p_t and the Lagrange multiplier λ^*. And finally, the lifetime budget constraint solves for the multiplier λ^* as a function of the entire sequence of prices $\boldsymbol{p} = \{p_t\}_{t\geq 0}$ after plugging in the two FOCs.

In summary, we can use the Foreign competitive equilibrium conditions to derive sequence-space representations of the Foreign consumption and labor supply functions, given by $c_t^* = \mathcal{C}_t^*(\boldsymbol{p})$

and $\ell_t^* = \mathcal{L}_t^*(\boldsymbol{p})$. The consumption function $\mathcal{C}_t^*(\boldsymbol{p})$ maps a time path of prices $\boldsymbol{p}$ into a level of consumption c_t^* at date t that is consistent the budget constraint and optimal behavior of the Foreign household. The labor supply function $\mathcal{L}_t^*(\boldsymbol{p})$ maps an infinite sequence of prices into the household's desired labor supply. Both of these functions are partial equilibrium objects. They are defined using only the conditions (47) and (49) describing the Foreign household problem and do not account for general equilibrium adjustments that work through market clearing conditions. With these two functions in hand, the intertemporal export supply function is simply given by $\mathcal{X}_t^*(\boldsymbol{p}) = A_t^* \mathcal{L}_t^*(\boldsymbol{p}) - \mathcal{C}_t^*(\boldsymbol{p})$, and its sequence-space Jacobian is characterized by

$$\frac{\partial \mathcal{X}_k^*}{\partial p_t} = A_k^* \frac{\partial \mathcal{L}_k^*}{\partial p_t} - \frac{\partial \mathcal{C}_k^*}{\partial p_t}. \tag{54}$$

We now show that this sequence-space Jacobian has a special form because Foreign households are permanent-income consumers. First, notice that plugging the labor supply and consumption functions into the Foreign household's labor-leisure condition implies

$$-u_{\ell\ell,k}^* \frac{\partial \mathcal{L}_k^*}{\partial p_t} = A_k^* u_{cc,k}^* \frac{\partial \mathcal{C}_k^*}{\partial p_t}$$

where we assume that $u(\cdot)$ is additively separable in consumption and labor supply. In other words, we can characterize the sequence-space Jacobian of the labor supply function in terms of that of the consumption function. Plugging back into the previous equation, we have

$$\frac{\partial \mathcal{X}_k^*}{\partial p_t} = -\left[(A_k^*)^2 \frac{u_{cc,k}^*}{u_{\ell\ell,k}^*} + 1 \right] \frac{\partial \mathcal{C}_k^*}{\partial p_t}.$$

Defining the relevant elasticities by

$$\gamma_t^* = -\frac{c_t^* u_{cc,t}^*}{u_{c,t}^*} \qquad \text{and} \qquad \phi_t^* = -\frac{\ell_t^* u_{\ell\ell,t}^*}{u_{\ell,t}^*}$$

we can rewrite this as

$$\frac{\partial \mathcal{X}_k^*}{\partial p_t} = -\left[(A_k^*)^2 \frac{\gamma_k^* u_{c,k}^*}{\phi_k^* u_{\ell,k}^*} \frac{\ell_k^*}{c_k^*} + 1 \right] \frac{\partial \mathcal{C}_k^*}{\partial p_t}.$$

Finally, using the Foreign labor-leisure condition, we arrive at

$$\frac{\partial \mathcal{X}_k^*}{\partial p_t} = \left(\frac{\gamma_k^*}{\phi_k^*} \frac{y_k^*}{c_k^*} - 1 \right) \frac{\partial \mathcal{C}_k^*}{\partial p_t}.$$

Two important takeaways emerge. First, just like in the endowment economy of Section 3.1, the intertemporal export supply elasticities can be micro-founded in terms of Foreigns intertemporal consumption function. Second, unlike in Section 3.1, the export response at date k to a price change at date t may no longer be negatively related to the consumption response $\partial \mathcal{C}_k^*$.

For example, under a standard CRRA calibration with $\gamma_t^* = \phi_t^* = 2$, the export response locally around balanced trade is actually 0. Under a trade surplus, with $y_k^* > c_k^*$, the export supply response is positively related to the consumption response. It is only under a trade deficit, with $y_k^* < c_k^*$, that the export supply response is negatively related to the consumption response as in Section 3.1.

C A New Keynesian Model

C.1 Preferences, Technologies and Resource Constraints

The world economy comprises two large countries, Home and Foreign, and each is populated by a representative household. There is a single final consumption good that is produced in and traded by both countries. We study the cashless limit, where a vanishing fraction of transactions in the two countries must be conducted in Home currency and Foreign currency, respectively. Finally, time is discrete, with $t \in \{0, 1, \ldots\}$, and we abstract from uncertainty.

Preferences. Households in both regions have direct preferences over the single consumption good, given by

$$V_0 = \sum_t \beta^t u(c_t, \ell_t) \qquad \text{and} \qquad V_0^* = \sum_t \beta^t u(c_t^*, \ell_t^*)$$

where c_t and c_t^* denote consumption in Home and Foreign. Households also supply labor, denoted ℓ_t and ℓ_t^*. Preferences are otherwise symmetric.

Technologies. Both countries are endowed with technologies to produce the single consumption good using labor. These are given by

$$y_t = A_t \ell_t \qquad \text{and} \qquad y_t^* = A_t^* \ell_t^*.$$

Labor is immobile across countries.

Resource constraints. The resource constraint for the consumption good at date t is given by

$$c_t + c_t^* = y_t + y_t^*.$$

C.2 Competitive Equilibrium

We now specify how households, firms, labor unions and the government interact in a world competitive equilibrium. We denote the numeraire price for the consumption good at date t by p_t, and the prices for Home and Foreign currency by $p_{H,t}$ and $p_{F,t}$, respectively. We will eventually choose Foreign currency as our numeraire and normalize $p_{F,t} = 1$, so that $p_{H,t}$ takes on the role of a nominal exchange rate, but we will for now keep all prices explicit. Finally, we denote by τ_t Home's import tariff at date t, which is the combination of a consumption tax and a production subsidy.

Asset markets. There are two nominal bonds. A unit of bond purchased at date t matures at date $t+1$ and pays one unit of local currency. We denote the prices at which the Home and Foreign bonds trade by $q_{H,t}$ and $q_{F,t}$, respectively. Financial markets are not segmented so that all households can trade both bonds.

Household problem. Households supply labor and earn the local wage w_t in Home and w_t^* in Foreign, in units of numeraire. In the cashless limit of our economy, the Home household budget constraint can be written as

$$q_{H,t}b_{H,t+1} + q_{F,t}b_{F,t+1} + (1+\tau_t)p_t c_t = p_{H,t}b_{H,t} + p_{F,t}b_{F,t} + w_t\ell_t + T_t.$$

And the Foreign household budget constraint is

$$q_{H,t}b^*_{H,t+1} + q_{F,t}b^*_{F,t+1} + p_t c^*_t = p_{H,t}b^*_{H,t} + p_{F,t}b^*_{F,t} + w^*_t\ell^*_t + T^*_t.$$

All prices are in units of numeraire. A unit of Home bond purchased at price $q_{H,t}$ at date t pays off one unit of Home currency at date $t+1$, whose numeraire value is $p_{H,t+1}$.

The problem of a household in Home is therefore to choose sequences $\{c_t, b_{H,t+1}, b_{F,t+1}, \ell_t\}$ to maximize V_0, subject to the Home budget constraint, taking as given prices $\{q_{g,t}, q_{s,t}, p_{c,t}, p_{g,t}, w_t\}$.

Firms. There is a representative firm in each country that operates the production technology and produces the consumption good using local labor. Firm profits are

$$\Pi_t = (1+\tau_t)p_t y_t - w_t\ell_t \qquad \text{and} \qquad \Pi^*_t = p_t y^*_t - w^*_t\ell^*_t.$$

The firm problem is to choose labor to maximize profits. It implies the first-order conditions

$$(1+\tau_t)p_t A_t = w_t \qquad \text{and} \qquad p_t A^*_t = w^*_t,$$

so that profits are 0.

Unions and labor market structure. We now introduce the key assumption: Wages are sticky in units of *local currency*. We follow the standard model of the wage rigidity literature Erceg et al. (2000). Household labor supply decisions are intermediated by labor unions. A household in Home supplies $\ell_{k,t}$ units of labor to $k \in [0,1]$ unions. Each labor union is endowed with a technology to differentiate these work hours into a differentiated labor factor k one-for-one. We abuse notation and directly denote this differentiated factor by $\ell_{k,t}$.

There is a firm in the economy which we call the labor packer. This firm demands differentiated labor factors $\ell_{k,t}$ and produces a pure intermediate input, which we call aggregate labor. It does so according to the production technology

$$\ell_t = \left(\int_0^1 \ell_{k,t}^{\frac{\epsilon-1}{\epsilon}} dk \right)^{\frac{\epsilon}{\epsilon-1}}.$$

This firm then sells aggregate labor as an input to the representative firm at price w_t in units of numeraire. The labor packer pays each union a price $w_{k,t}$ for its differentiated labor variety. Cost minimization by the labor packer therefore yields a demand function

$$\ell_{k,t} = \left(\frac{w_{k,t}}{w_t} \right)^{-\epsilon} \ell_t$$

and an aggregate wage index

$$w_t = \left(\int_0^1 w_{k,t}^{1-\epsilon} dk \right)^{\frac{1}{1-\epsilon}}.$$

Union k chooses its wage $\{w_{k,t}\}$ to maximize the lifetime utility of the representative household. We model nominal rigidities as follows. At each date t, the union can change its wage $w_{k,t}$. But when the new *local currency wage* deviates from the previous one, the union must pay an adjustment cost that is passed to the household directly as a utility cost. We refer to *local currency wages* as wages in units of local currency rather than numeraire. Home local currency wages are given by

$$W_{k,t} \equiv \frac{w_{k,t}}{p_{H,t}}$$

and we define their growth rate as

$$1 + \pi_{k,t}^W = \frac{W_{k,t}}{W_{k,t-1}}.$$

We can therefore associate the union problem with the Lagrangian

$$L = \sum_t \beta^t \left[u\left(c_t(\{W_{k,t}\}_k), \int_0^1 \ell_{k,t} dk \right) - \frac{\delta}{2} \frac{1}{1+\pi_t^W} \int_0^1 (\pi_{k,t}^W)^2 dk \right] + \sum_t \beta^t \lambda_t \left[\frac{W_{k,t}}{W_{k,t-1}} - (1+\pi_{k,t}^W) \right]$$

subject to the demand function for $\ell_{k,t}$, which we simply plug in directly. We scale the adjustment cost by $\frac{1}{1+\pi_t^W}$, which will allow us to derive a particularly tractable wage Phillips curve. The first-order condition for $W_{k,t}$ is

$$0 = u_{c,t} \frac{\partial c_t}{\partial W_{k,t}} + u_{\ell,t} \frac{\partial \ell_{k,t}}{\partial W_{k,t}} + \frac{1}{W_{k,t-1}} \lambda_t - \beta \lambda_{t+1} W_{k,t+1} \frac{1}{W_{k,t}} \frac{1}{W_{k,t}}$$

where

$$\frac{\partial \ell_{k,t}}{\partial W_{k,t}} = -\epsilon \left(\frac{w_{k,t}}{w_t} \right)^{-\epsilon-1} \frac{1}{w_t} \ell_t$$

The second optimality condition for $\pi^W_{k,t}$ is

$$0 = -\delta \frac{1}{1+\pi^W_t} \pi^W_t - \lambda_t.$$

To derive the wage Phillips curve, we have to work out how wage changes affect the Home household. This requires expressing the household problem in units of local currency. We can first rewrite the Home household budget constraint in units of numeraire, accounting for the fact that the household supplies labor to different unions,

$$q_{H,t} b_{H,t+1} + q_{F,t} b_{F,t+1} + (1+\tau_t) p_t c_t = p_{H,t} b_{H,t} + p_{F,t} b_{F,t} + \int_0^1 (1+\tau^\ell) w_{k,t} \ell_{k,t} dk + T_t,$$

where τ^ℓ is an employment subsidy on union k's wage payment. When union k pays the household a total compensation $w_{k,t}\ell_{k,t}$ for her labor, then the government pays an additional proportional subsidy $\tau^\ell w_{k,t}\ell_{k,t}$. At the same time, the government pays for this subsidy by taxing households lump-sum on aggregate labor. But unions take the aggregate government rebate T_t as given. Rewriting the budget constraint in local currency units yields

$$\frac{q_{H,t}}{p_{H,t}} b_{H,t+1} + \frac{q_{F,t}}{p_{H,t}} b_{F,t+1} + (1+\tau_t) \frac{p_t}{p_{H,t}} c_t = b_{H,t} + \frac{p_{F,t}}{p_{H,t}} b_{F,t} + (1+\tau^\ell) \int_0^1 W_{k,t} \left(\frac{W_{k,t}}{W_t} \right)^{-\epsilon} \ell_t dk + \frac{1}{p_{H,t}} T_t.$$

Finally, we apply the envelope theorem and obtain the income effect on consumption that the union internalizes when changing wages as

$$\frac{\partial c_t}{\partial W_{k,t}} = \frac{p_{H,t}}{(1+\tau_t) p_t} (1+\tau^\ell)(1-\epsilon) \left(\frac{W_{k,t}}{W_t} \right)^{-\epsilon} \ell_t.$$

Finally, we initialize all unions at the symmetric wage distribution $w_{k,0} = w_{k',0}$ which implies that unions will always be symmetric ex post as well under Rotemberg adjustment costs, with $w_{k,t} = w_t$ for all k and t. Leveraging this, we can then write the two first-order conditions of the union problem as

$$\begin{aligned} 0 &= u_{c,t} \frac{p_{H,t}}{(1+\tau_t) p_t} (1+\tau^\ell)(1-\epsilon) \ell_t - u_{\ell,t} \epsilon \frac{1}{W_t} \ell_t + \frac{1}{W_{t-1}} \lambda_t - \beta \lambda_{t+1} W_{t+1} \frac{1}{W_t} \frac{1}{W_t} \\ 0 &= -\delta \frac{1}{1+\pi^W_t} \pi^W_t - \lambda_t. \end{aligned}$$

From the second condition, we have $\lambda_{t+1} = -\delta \frac{\pi^W_{t+1}}{1+\pi^W_{t+1}}$. And plugging this into the first condition yields

$$0 = u_{c,t} \frac{w_t}{(1+\tau_t) p_t} (1+\tau^\ell)(1-\epsilon) \ell_t - u_{\ell,t} \epsilon \ell_t - \delta \pi^W_t + \delta \beta \pi^W_{t+1}$$

or simply

$$\pi_t^W = -\frac{\epsilon}{\delta}\left[u_{c,t}\frac{w_t}{(1+\tau_t)p_t}(1+\tau^\ell)\frac{\epsilon-1}{\epsilon} + u_{\ell,t}\right]\ell_t + \beta\pi_{t+1}^W.$$

This is the exact analog of the continuous time Phillips curve we later derive in Section D. If we had not scaled the Rotemberg adjustment cost by $\frac{1}{1+\pi_t^W}$, we would have obtain the same Phillips curve as in Auclert et al. (2024b),

$$\pi_t^W(1+\pi_t^W) = -\frac{\epsilon}{\delta}\left[u_{c,t}\frac{w_t}{(1+\tau_t)p_t}(1+\tau^\ell)\frac{\epsilon-1}{\epsilon} + u_{\ell,t}\right]\ell_t + \beta\pi_{t+1}^W(1+\pi_{t+1}^W).$$

Foreign wage Phillips curve. Crucially, Foreign wages are sticky in Foreign currency rather than Home currency. So we define the local currency wage in Foreign as

$$W_t^* = \frac{w_t^*}{p_{F,t}}.$$

Most of the derivation of the Foreign Phillips curve is symmetric. The income effect on consumption of a household in Foreign is therefore given by

$$\frac{\partial c_t^*}{\partial W_{k,t}^*} = \frac{p_{F,t}}{p_t}(1+\tau^\ell)(1-\epsilon)\left(\frac{W_{k,t}^*}{W_t^*}\right)^{-\epsilon}\ell_t^*.$$

We define Foreign inflation as

$$\pi_t^{W,*} = \frac{W_t^*}{W_{t-1}^*} - 1 = \frac{p_{F,t-1}}{w_{t-1}^*}\frac{w_t^*}{p_{F,t}} - 1 = \frac{1+\pi_t^{w,*}}{1+\pi_t^F} - 1.$$

This implies a symmetric Phillips curve.

Policy. Fiscal policy in Home comprises four instruments: the import tariff τ_t, the lump-sum rebate T_t, and the employment subsidy τ^ℓ. We assume that the Home planner can set tariffs optimally but must run a balanced budget each period. This requires

$$\tau_t p_t C_t = T_t + \tau_t p_t y_t + \tau^\ell w_t \ell_t,$$

where on the LHS we have tariff revenue and on the RHS total outlays, which comprise the transfer T_t, the production subsidy to firms, and the employment subsidy to unions.

The Home planner sets monetary policy, i.e., the Home bond price $q_{H,t}$, optimally. Monetary policy in Foreign sets the Foreign bond price according to a Taylor rule,

$$q_{F,t} = \mathcal{T}_t(\boldsymbol{\pi}^{W,*}).$$

The Taylor rule $\mathcal{T}_t$ allows for a flexible dependence of the date t bond price on Foreign nominal wage inflation $\boldsymbol{\pi}^{W,*} = \{\pi_k^{W,*}\}_{k\geq 0}$.

Markets and equilibrium. Five markets must clear in equilibrium. The world goods market coincides with the resource constraint introduced earlier. Labor market clearing in both countries is already implicit in our notation, where ℓ_t and ℓ_t^* were used to denote both labor demand and supply. Finally, asset market clearing requires

$$b_{H,t} + b_{H,t}^* = 0 \qquad \text{and} \qquad b_{F,t} + b_{F,t}^* = 0.$$

Under no-arbitrage, households' portfolio positions in the two bonds is indeterminate. The two asset market clearing conditions are thus effectively equivalent to a single market clearing condition in terms of total asset holdings or wealth. In particular, we can multiply the two bond market clearing conditions by the bond prices, add them up, and arrive at which by adding simply implies

$$B_t + B_t^* = 0,$$

where we define $B_t = q_{H,t-1}b_{H,t} + q_{F,t-1}b_{F,t}^*$ and $B_t^* = q_{H,t-1}b_{H,t}^* + q_{F,t-1}b_{F,t}^*$ as total wealth in units of numeraire. We will drop this market clearing condition by Walras' law and instead work with goods market clearing.

Definition 5 (World Competitive Equilibrium)**.** *Taking as given exogenous sequences of shocks $(\boldsymbol{A}, \boldsymbol{A}^*)$ and Home policy $(\boldsymbol{\tau}, \boldsymbol{q}_H)$, a world competitive equilibrium with sticky wages comprises a Home allocation $(\boldsymbol{y}, \boldsymbol{c}, \boldsymbol{\ell})$, a Foreign allocation $(\boldsymbol{y}^*, \boldsymbol{c}^*, \boldsymbol{\ell}^*)$, and prices $(\boldsymbol{q}_F, \boldsymbol{p}, \boldsymbol{p}_H, \boldsymbol{p}_F, \boldsymbol{w}, \boldsymbol{w}^*)$ such that households, unions and firms in both countries optimize and markets clear.*

C.3 Optimal Monetary and Tariff Policy

The Home planning problem is to choose monetary and tariff policy $(\boldsymbol{\tau}, \boldsymbol{q}_H)$ to maximize the lifetime utility of the representative household, taking as given all the conditions that characterize world competitive equilibrium. We now present the main result of this Appendix and then provide a constructive proof.

Proposition 6 (Optimal monetary and tariff policy)**.** *Home's jointly optimal monetary and tariff policy restores the flexible-wage allocation and sets import tariffs according to the intertemporal tariff formula* (13).

To prove Proposition 6, we start with a characterization of the equilibrium and implementability conditions. The problem of the Home household gives rise to the consumption Euler equation

$$u_{c,t} = \frac{p_{H,t+1}}{q_{H,t}} \frac{(1+\tau_t)p_t}{(1+\tau_{t+1})p_{t+1}} \beta u_{c,t+1},$$

as well as the usual no-arbitrage condition, according to which returns on both bonds must be equalized in units of numeraire. This implies uncovered interest parity (UIP),

$$\frac{q_{F,t}}{p_{F,t+1}} = \frac{q_{H,t}}{p_{H,t+1}}.$$

The complete list of equilibrium conditions is therefore given by

$$\begin{aligned}
y_t &= A_t \ell_t \\
y_t^* &= A_t^* \ell_t^* \\
y_t + y_t^* &= c_t + c_t^* \\
0 &= B_t + B_t^* \\
(1+\tau_t) p_t A_t &= w_t \\
p_t A_t^* &= w_t^* \\
u_{c,t} &= \frac{p_{H,t+1}}{q_{H,t}} \frac{(1+\tau_t) p_t}{(1+\tau_{t+1}) p_{t+1}} \beta u_{c,t+1} \\
u_{c,t}^* &= \frac{p_{F,t+1}}{q_{F,t}} \frac{p_t}{p_{t+1}} \beta u_{c,t+1}^* \\
\frac{q_{F,t}}{p_{F,t+1}} &= \frac{q_{H,t}}{p_{H,t+1}} \\
B_{t+1} + (1+\tau_t) p_t c_t &= \frac{p_{H,t}}{q_{H,t-1}} B_t + w_t \ell_t + T_t \\
B_{t+1}^* + p_t c_t^* &= \frac{p_{F,t}}{q_{F,t-1}} B_t^* + w_t^* \ell_t^* \\
\pi_t^W &= -\frac{\epsilon}{\delta} \left[u_{c,t} \frac{w_t}{(1+\tau_t) p_t} (1+\tau^\ell) \frac{\epsilon - 1}{\epsilon} + u_{\ell,t} \right] \ell_t + \beta \pi_{t+1}^W \\
\pi_t^{W,*} &= -\frac{\epsilon}{\delta} \left[u_{c,t}^* \frac{w_t^*}{p_t} (1+\tau^\ell) \frac{\epsilon - 1}{\epsilon} + u_{\ell,t}^* \right] \ell_t^* + \beta \pi_{t+1}^{W,*} \\
\pi_t^W &= \frac{p_{H,t-1}}{w_{t-1}} \frac{w_t}{p_{H,t}} - 1 \\
\pi_t^{W,*} &= \frac{p_{F,t-1}}{w_{t-1}^*} \frac{w_t^*}{p_{F,t}} - 1 \\
T_t &= \tau_t p_t (c_t - y_t) \\
q_{F,t} &= \mathcal{T}_t(\boldsymbol{\pi}^{W,*})
\end{aligned}$$

where we have already subsumed that part of the fiscal rebate that pays for the employment subsidy. We therefore have 17 equations in the 17 unknowns

$$\left\{y_t, y_t^*, \ell_t, \ell_t^*, c_t, c_t^*, B_t, B_t^*, T_t, p_t, w_t, w_t^*, p_{H,t}, p_{F,t}, \pi_t^W, \pi_t^{W,*}, q_{F,t}\right\}$$

as functions of the exogenous sequences

$$\left\{A_t, A_t^*, \tau_t, q_{H,t}\right\}.$$

From here, we get to drop one equation due to Walras' law, and we still have to make one price normalization by picking a numeraire. This implies that we get to 16 equations in 16 unknowns.

We now reduce the equations the Home planner must respect as follows. We plug in for y_t and y_t^* using the production functions, which yields the world goods market clearing condition in terms of consumption and labor,

$$A_t \ell_t + A_t^* \ell_t^* = c_t + c_t^*.$$

Next, we use the firm optimality conditions to solve out for wages in the two Phillips curves, and we rewrite them as follows. For Home, we have

$$-\frac{u_{\ell,t}}{u_{c,t}} \frac{1}{A_t} = 1 - \Lambda_t^{\ell}$$

where the Home aggregate labor wedge is defined as

$$\Lambda_t^{\ell} = 1 - (1 + \tau^{\ell}) \frac{\epsilon - 1}{\epsilon} - \frac{\epsilon}{\delta} \frac{1}{A_t \ell_t u_{c,t}} (\pi_t^W - \beta \pi_{t+1}^W)$$

For Foreign, we have

$$-\frac{u_{\ell,t}^*}{u_{c,t}^*} \frac{1}{A_t^*} = 1 - \Lambda_t^{\ell,*}$$

where the Foreign aggregate labor wedge is defined as

$$\Lambda_t^{\ell,*} = 1 - (1 + \tau^{\ell,*}) \frac{\epsilon - 1}{\epsilon} - \frac{\epsilon}{\delta} \frac{1}{A_t^* \ell_t^* u_{c,t}^*} (\pi_t^{W,*} - \beta \pi_{t+1}^{W,*}).$$

In other words, we see that the wage Phillips curve put a wedge between the marginal rates of substitution between consumption and labor, $-\frac{u_{\ell,t}}{u_{c,t}}$, and the marginal rates of transformation between consumption and labor, A_t. These labor wedges have a constant component, the markup distortion due to monopolistic competition, and a time-varying component due to non-zero wage inflation.

Plugging the government budget constraint into the two household budget constraints, we arrive at the country-level laws of motion for net foreign asset (NFA) positions,

$$B_{t+1} = \frac{p_{H,t}}{q_{H,t-1}} B_t + p_t(A_t \ell_t - c_t)$$
$$B^*_{t+1} = \frac{p_{F,t}}{q_{F,t-1}} B^*_t + p_t(A^*_t \ell^*_t - c^*_t).$$

We see from here that Walras law must hold, since the goods market clearing condition will imply that $B_{t+1} + B^*_{t+1} = 0$, as long as the initial condition satisfies $B_0 + B^*_0 = 0$.

Finally, we drop the asset market clearing condition due to Walras' law, we drop the NFA laws of motion since NFA does not appear in other equations, and we choose Foreign currency as our numeraire, setting $p_{F,t} = 1$ for all t. This implies that $p_{H,t}$ is the relative price (exchange rate) between Home and Foreign currencies.

Foreign can therefore be summarized by the following conditions:

$$u^*_{c,t} = \frac{1}{q_{F,t}} \frac{p_t}{p_{t+1}} \beta u^*_{c,t+1}$$
$$-\frac{u^*_{\ell,t}}{u^*_{c,t}} \frac{1}{A^*_t} = 1 - \Lambda^{\ell,*}_t$$
$$\Lambda^{\ell,*}_t = 1 - (1 + \tau^{\ell,*}) \frac{\epsilon - 1}{\epsilon} - \frac{\epsilon}{\delta} \frac{1}{A^*_t \ell^*_t u^*_{c,t}} (\pi^{W,*}_t - \beta \pi^{W,*}_{t+1})$$
$$\pi^{W,*}_t = \frac{p_t}{p_{t-1}} \frac{A^*_t}{A^*_{t-1}} - 1$$
$$q_{F,t} = \mathcal{T}_t(\boldsymbol{\pi}^{W,*})$$

This block of equations now implies a representation for Foreign export supply in terms of an intertemporal export supply function,

$$x^*_t = A^*_t \ell^*_t - c^*_t = \mathcal{X}^*_t(\boldsymbol{p}).$$

Taking as given a sequence of world prices $\boldsymbol{p}$, the firm optimality condition pins down Foreign wages and therefore wage inflation $\boldsymbol{\pi}^{W,*}$. Wage inflation pins down Foreign monetary policy in terms of the bond price sequence $\boldsymbol{q}_F$ via the Taylor rule. On the household side, this pins down Foreign consumption $\boldsymbol{c}^*$. Finally, from the Phillips curve we get the aggregate labor wedges $\boldsymbol{\Lambda}^{\ell,*}$ and labor supply $\boldsymbol{\ell}^*$.

We are consequently left with the following set of conditions that the Home planner must respect,

$$
\begin{aligned}
\mathcal{X}_t^*(\boldsymbol{p}) &= c_t - A_t \ell_t \\
u_{c,t} &= \frac{p_{H,t+1}}{q_{H,t}} \frac{1+\tau_t}{1+\tau_{t+1}} \frac{p_t}{p_{t+1}} \beta u_{c,t+1} \\
q_{H,t} &= p_{H,t+1} \mathcal{Q}_t^*(\boldsymbol{p}) \\
-\frac{u_{\ell,t}}{u_{c,t}} \frac{1}{A_t} &= 1 - \Lambda_t^\ell \\
\Lambda_t^\ell &= 1 - (1+\tau^\ell) \frac{\epsilon - 1}{\epsilon} - \frac{\epsilon}{\delta} \frac{1}{A_t \ell_t u_{c,t}} (\pi_t^W - \beta \pi_{t+1}^W) \\
\pi_t^W &= \frac{1+\tau_t}{1+\tau_{t-1}} \frac{p_t}{p_{t-1}} \frac{A_t}{A_{t-1}} \frac{p_{H,t-1}}{p_{H,t}} - 1
\end{aligned}
$$

where

$$
q_{F,t} = \mathcal{Q}_t^*(\boldsymbol{p})
$$

represents an intertemporal sequence-space representation for Foreign monetary policy, which also follows from above. In other words, given Home policy $(\boldsymbol{\tau}, \boldsymbol{q}_H)$, the above 6 equations solve for the remaining 6 unknowns $(\boldsymbol{c}, \boldsymbol{\ell}, \boldsymbol{p}, \boldsymbol{p}_H, \boldsymbol{\Lambda}^\ell, \boldsymbol{\pi}^W)$.

We can rewrite the Home Euler equation as

$$
u_{c,t} = \beta \frac{1}{\mathcal{Q}_t^*(\boldsymbol{p})} \frac{p_t}{p_{t+1}} \frac{1+\tau_t}{1+\tau_{t+1}} u_{c,t+1}.
$$

This implies that the Home bond price $q_{H,t}$ only appears in the UIP condition. Therefore, for any given price sequence $\boldsymbol{p}$, the Home planner can implement any desired Home currency price sequence $\boldsymbol{p}_H$ by setting monetary policy appropriately. And we can drop UIP as an implementability condition.

Next, we use the Euler equation to solve out for cum-tariff prices in the definition of wage inflation, which yields

$$
\pi_t^W = \frac{A_t}{A_{t-1}} \frac{p_{H,t-1}}{p_{H,t}} \frac{u_{c,t}}{\beta u_{c,t+1}} \mathcal{Q}_t^*(\boldsymbol{p}) - 1.
$$

As a result, the import tariff $\boldsymbol{\tau}$ now only appears in the consumption Euler equation. This implies that, for any given price sequence $\boldsymbol{p}$ and long-run consumption level c_{ss}, the planner can pick import tariffs $\boldsymbol{\tau}$ to implement a desired consumption sequence $\boldsymbol{c}$. Since tariffs no longer appear in any of the other conditions, we can therefore drop the Euler equation as an implementability condition. We therefore arrive at the following characterization of implementability.

Lemma 3 (Implementability)**.** *Taking as given TFP sequence $\boldsymbol{A}$, as well as the intertemporal export supply* *and monetary policy functions for Foreign, $\mathcal{X}^*(\boldsymbol{p})$ and $\mathcal{Q}^*(\boldsymbol{p})$, sequences $(\boldsymbol{c}, \boldsymbol{\ell}, \boldsymbol{p}, \boldsymbol{p}_H, \boldsymbol{\Lambda}^\ell, \boldsymbol{\pi}^W)$ form part of a world competitive equilibrium if and only if*

$$\mathcal{X}_t^*(\boldsymbol{p}) = c_t - A_t \ell_t$$

$$-\frac{u_{\ell,t}}{u_{c,t}} \frac{1}{A_t} = 1 - \Lambda_t^\ell$$

$$\Lambda_t^\ell = 1 - (1 + \tau^\ell) \frac{\epsilon - 1}{\epsilon} - \frac{\epsilon}{\delta} \frac{1}{A_t \ell_t u_{c,t}} (\pi_t^W - \beta \pi_{t+1}^W)$$

$$\pi_t^W = \frac{A_t}{A_{t-1}} \frac{p_{H,t-1}}{p_{H,t}} \frac{u_{c,t}}{\beta u_{c,t+1}} \mathcal{Q}_t^*(\boldsymbol{p}) - 1.$$

We are now ready to characterize Home's optimal monetary and tariff policy. Notice that the Home currency price $\boldsymbol{p}_H$ only appears in the determination of wage inflation $\boldsymbol{\pi}^W$. Using monetary policy, the planner can therefore implement any desired sequence of wage inflation. Dropping the fourth equation as an implementability condition, notice that wage inflation then only appears in the definition of the labor wedge. Therefore, the Home planner can pick any desired sequence of labor wedges $\boldsymbol{\Lambda}^\ell$, by appropriately choosing wage inflation and monetary policy. We can thus also drop the third equation as an implementability condition. This leaves us with the first two equations. Finally, by the same argument, notice that the labor wedge now only appears in the second equation. And since the planner can choose it freely, we can also drop the sequence equation as an implementability condition.

We have therefore arrived at a characterization of implementability conditions in terms of just

$$\mathcal{X}_t^*(\boldsymbol{p}) = c_t - A_t \ell_t,$$

exactly as in Section 2.3. This implies that the solution to the Home planning problem will satisfy the efficiency condition (12) for intertemporal terms of trade, as well as the classic MRS = MRT efficiency condition between consumption and labor. The latter implies that

$$\Lambda_t^\ell = 0$$

at the optimal allocation for all t. This concludes the proof.

C.4 Discussion

Proposition 6 shows that the Home planner can implement the efficient allocation with monetary and tariff policy. Intuitively, the planner has two targets and exactly two instruments. The targets are (1) closing the labor wedge, i.e., production efficiency, and (2) terms of trade manipulation. The Home planner uses monetary policy to close the labor wedge, and then sets import tariffs to target the desired intertemporal terms of trade. Optimal monetary policy is to ensure nominal wage stability.

If monetary policy was constrained, the planner would be unable to implement the efficient allocation. This would put us in the environment of Section 2.5. The optimal second-best tariff would then trade off terms of trade manipulation against production efficiency.

When monetary policy can be set optimally to achieve Home production efficiency, however, our intertemporal tariff formula (13) directly applies to this economy and characterizes the optimal import tariff. Intertemporal export supply elasticities, summarized by the export supply function $\mathcal{X}^*(\boldsymbol{p})$ and its log sequence-space Jacobian $\boldsymbol{\mathcal{E}}$, are a sufficient statistic for the optimal tariff, together with tariff revenue weights. The determination of Foreign's intertemporal export supply elasticities in this New Keynesian model is different from the environment of Section 3.1 for two reasons: First, we here allow for production, so that export supply is now governed both by household consumption demand and labor supply,

$$\frac{\partial \mathcal{X}_k^*}{\partial p_t} = A_k^* \frac{\partial \mathcal{L}_k^*}{\partial p_t} - \frac{\partial \mathcal{C}_k^*}{\partial p_t}.$$

Second, the Foreign economy now features distortions in the form of nominal wage rigidity. This implies that the Foreign export supply function $\mathcal{X}_t^*(\boldsymbol{p})$ is shaped by local business cycle conditions in Foreign. The Home planner exploits this, internalizing that intertemporal export supply elasticities now interact with business cycle conditions. Also, whereas in Section 3.1 we derived $\mathcal{X}_t^*(\boldsymbol{p})$ as a partial equilibrium object, representing households' decision problems given prices $\boldsymbol{p}$, the intertemporal export supply function is a general equilibrium object here. It encodes the general equilibrium determination of Foreign labor wedges $\boldsymbol{L}^{\ell,*}$, Foreign wage inflation $\boldsymbol{\pi}^{W,*}$ via the Phillips curve, and Foreign monetary policy $\boldsymbol{q}_F$ via the Taylor rule.

D Quantitative Appendix

We work in continuous time to characterize and numerically solve the Ramsey problem for optimal second-best tariffs. In this Appendix, we briefly recast the quantitative model of Section 5 in continuous time. Then we characterize implementability conditions for the Ramsey problem. We define the standard Ramsey problem as well as the augmented timeless one.

D.1 Recursive Representation of the Household Problem

The problem of a household living in Home at date t, with wealth a and individual productivity z, can be written recursively in continuous time in terms of the Hamilton-Jacobi-Bellman (HJB) equation

$$\rho V_t(a,z) = u(c_t(a,z)) - v(\ell_t) + (r_t a + e_t(z) - c_t(a,z))\partial_a V_t(a,z) + \lambda(V_t(a,z') - V_t(a,z)) + \partial_t V_t(a,z).$$

This equation holds in the interior of the state space $(a, z) \in [\underline{a}, \infty) \times \{z^L, z^H\}$. On the boundary at $\underline{a}$, the value function is characterized by the usual state constrained boundary condition

$$u'(r_t \underline{a} + e_t(z)) \leq \partial_a V_t(\underline{a}, z)$$

for all z. For convenience, we also introduce the shorthand notation

$$s_t(a, z) = r_t a + e_t(z) - c_t(a, z)$$

for the household's savings policy function.

The household takes as given a sequence of real interest rates r_t, which we discuss further below, as well as the union-intermediated sequence of labor supply ℓ_t. As in the main text, we write the household problem directly in terms of total real wealth, implicitly subsuming the portfolio choice problem between Home and Foreign bonds, which gives rise to the usual no-arbitrage (UIP) condition. Finally, recall that wealth a here is in units of Home CPI (in Home currency), which we define next.

Consumption. The consumption policy function $c_t(a, z)$ is implicitly defined by the first-order condition

$$u'(c_t(a, z)) = \partial_a V_t(a, z).$$

Given a rate of real consumption, the household's demand functions are given by

$$c_{j,t}(a, z) = \alpha_j \frac{P_{j,t}}{P_t} c_t(a, z) \qquad \text{and} \qquad c_{j\omega,t}(a, z) = \theta_{j\omega} \left(\frac{(1 + \tau_{j\omega,t}) p_{j\omega,t}}{P_{j,t}} \right)^{-\eta_j} c_{j,t}(a, z),$$

where $c_{j,t}(a, z)$ denotes consumption of good j, which itself is a bundle of varieties $c_{j\omega,t}$. We denote by $p_{j\omega,t}$ the Home-currency price of good j variety ω at date t. We denote by $\theta_{j\omega}$ the Home household's CES weight on good j's variety ω, which by assumption is simply $\theta_{jH} = 1 - \theta_j$ and $\theta_{jF} = \theta_j$. All households in Home face the same consumer price index (CPI) defined in units of Home currency as

$$P_t = \sum_j \alpha_j \log \frac{P_{j,t}}{\alpha_j} \qquad \text{where} \qquad P_{j,t} = \left(\sum_\omega \theta_{j\omega} ((1 + \tau_{j\omega,t}) p_{j\omega,t})^{1-\eta_j} \right)^{\frac{1}{1-\eta_j}}$$

is the effective price of good j across varieties.

Income. The household's real income is given by

$$e_t(z) = z \frac{W_t \ell_t + T_t + \Pi_t}{P_t},$$

where we assume that all income is proportional to the household's individual productivity z, which follows a two-state Markov chain with $z \in \{z^L, z^H\}$ and transition rate λ. We denote by $w_t = \frac{W_t}{P_t}$ the real wage rate in Home, which we further discuss below. T_t is a lump-sum government rebate in units of Home currency, and Π_t represent total corporate profits in Home.

Foreign. We characterize the recursive representation of the Foreign household problem in terms of the state variables (a, z), where a here refers to total wealth in units of Foreign CPI (in Foreign currency). The problem of a household living in Foreign at date t, with wealth a and individual productivity z, can be represented by the HJB

$$\rho V_t^*(a,z) = u(c_t^*(a,z)) - v(\ell_t^*) + (r_t^* a + e_t^*(z) - c_t^*(a,z))\partial_a V_t^*(a,z) + \lambda(V_t^*(a,z') - V_t^*(a,z)) + \partial_t V_t^*(a,z).$$

It is therefore completely symmetric to the Home household problem, except that the household takes as given labor supply ℓ_t^*, real interest rates r_t^*, and income

$$e_t^*(z) = z\frac{W_t^*\ell_t^* + T_t^* + \Pi_t^*}{P_t^*},$$

which differ between Home and Foreign.

The Foreign household's demand functions are given by

$$c_{j,t}^*(a,z) = \alpha_j^* \frac{P_{j,t}^*}{P_t^*} c_t^*(a,z) \qquad \text{and} \qquad c_{j\omega,t}^*(a,z) = \theta_{j\omega}^* \left(\frac{p_{j\omega,t}^*}{P_{j,t}^*}\right)^{-\eta_j^*} c_{j,t}^*(a,z),$$

where $p_{j\omega,t}^* = \frac{p_{j\omega,t}}{\mathcal{E}_t}$ denotes the Foreign-currency price of good j variety ω and the exchange rate $\mathcal{E}_t$ maps the Home-currency price to the Foreign-currency price. Foreign's CPI is defined as

$$P_t^* = \sum_j \alpha_j^* \log \frac{P_{j,t}^*}{\alpha_j^*} \qquad \text{where} \qquad P_{j,t}^* = \left(\sum_\omega \theta_{j\omega}^* (p_{j\omega,t}^*)^{1-\eta_j^*}\right)^{\frac{1}{1-\eta_j^*}}.$$

D.2 Kolmogorov Forward Equations

We denote Home and Foreign's joint densities over household wealth and labor productivities at date t by $g_t(a,z)$ and $g_t^*(a,z)$. We also refer to these as the cross-sectional household distributions.

Taking as given initial distributions $g_0(a,z)$ and $g_0^*(a,z)$, their laws of motion are characterized by Kolmogorov forward equations. For Home, we have

$$\partial_t g_t(a,z) = -\partial_a\Big[s_t(a,z)g_t(a,z)\Big] + \lambda\Big[g_t(a,z') - g_t(a,z)\Big],$$

and for Foreign we have symmetrically

$$\partial_t g_t^*(a,z) = -\partial_a\Big[s_t^*(a,z)g_t^*(a,z)\Big] + \lambda\Big[g_t^*(a,z') - g_t^*(a,z)\Big].$$

This leaves the question of how the initial distributions $g_0(a,z)$ and $g_0^*(a,z)$ are determined. We discuss this next.

D.3 Initial Distributions and Revaluation Effects

We characterize the household problems and the joint densities in terms of real wealth in units of country-specific CPI. At time 0, prices in this economy may jump either because the Ramsey planner announces a new Ramsey plan or because of the realization of an MIT shock. In other words, the distribution is not predetermined and therefore pinned down at date 0 in terms of real wealth. Instead, it is predetermined in terms of households' holdings of physical asset shares.

We start by considering Home. Denote b_H and b_F the number of Home and Foreign bonds held by a household in Home. With bonds instantly maturing, the Home bond pays 1 unit of domestic currency and the Foreign bond pays 1 unit of Foreign currency, which is converted into $\mathcal{E}_0$ units of Home currency. Finally, we deflate by Home CPI, so that $a = \frac{b_H + \mathcal{E}_0 b_F}{P_0}$. Now denote the initial joint density over asset positions by $\tilde{g}(b_H, b_F, z)$.

We now obtain the appropriate initial condition for the joint density over real wealth and labor productivities via a change of variables. Let $\xi : \mathbb{R}^2 \to \mathbb{R}$ be the function $\xi(b_H, b_F) = \frac{b_H + \mathcal{E}_0 b_F}{P_0}$. Then for any Borel set $A \in \mathbb{R}$ and any z, we define g_0 via

$$\int_A g_0(a,z)da = \iint_{\xi^{-1}(A)} \tilde{g}(b_H, b_F, z)\, db_H\, db_F.$$

So g_0 is the unique density that makes the wealth-density match the asset-share-density. Another and more intuitive way to write this is simply as

$$g_0(a,z) = \iint_{\mathbb{R}^2} \tilde{g}(b_H, b_F, z) \cdot \delta\left(a - \frac{b_H + \mathcal{E}_0 b_F}{P_0}\right) db_H\, db_F,$$

where $\delta(\cdot)$ is the Diract delta function. Intuitively, the RHS just assigns all mass that corresponds to states where real wealth is equal to a given asset shares to $g_0(a,z)$. Notice that this mapping encodes the relevant revaluation effects that may occur at date 0 because of either exchange rate or CPI jumps.

The argument for finding the initial distribution $g_0^*(a,z)$ is symmetric.

D.4 Labor Unions

We now give a detailed description of our model's labor market structure, focusing on Home as Foreign is symmetric. Households supply labor to each of $k \in [0,1]$ labor unions. A household's total hours of work are $\ell_t = \int_0^1 \ell_{k,t} dk$. Each union pays the nominal wage $W_{k,t}$ in units of Home-currency CPI. The real income term $e_t(z)$ in the HJB of Appendix D.1 therefore becomes

$$e_t(z) = z\frac{1}{P_t}\left(\int_0^1 W_{k,t}\ell_{k,t}dk + \Pi_t + T_t\right)$$

Union $k \in [0,1]$ transforms hours into a differentiated labor service according to the aggregation technology

$$L_{k,t} = \iint z\ell_{k,t}g_t(a,z)\,da\,dz,$$

where $L_{k,t}$ is in units of effective labor.

Rationing. As is standard practice in the wage rigidity literature (Erceg et al., 2000; Auclert et al., 2024b), unions ration labor across households, so that all households work the same hours. This implies $L_{k,t} = \ell_{k,t}\iint zg_t(a,z)\,da\,dz = \ell_{k,t}$, where we normalize $\iint zg_t(a,z)\,da\,dz = 1$ as in the main text.

Labor packer. Unions sell their differentiated labor services to a labor packer. This packer operates the CES aggregation technology

$$\ell_t = \left(\int_0^1 L_{k,t}^{\frac{\epsilon^w-1}{\epsilon^w}}\,dk\right)^{\frac{\epsilon^w}{\epsilon^w-1}},$$

where ϵ^w is the elasticity of substitution between differentiated labor inputs. The labor packer then sells this homogeneous labor factor to the firms. The standard cost minimization problem implies

$$L_{k,t} = \left(\frac{W_{k,t}}{W_t}\right)^{-\epsilon^w}\ell_t \qquad \text{and} \qquad W_t = \left(\int_0^1 W_{k,t}^{1-\epsilon^w}\,dk\right)^{\frac{1}{1-\epsilon}}.$$

Labor supply schedule. Wages are fully flexible. We assume that union k chooses the sequence of nominal wages $W_{k,t}$ to maximize the objective

$$\max_{W_{k,t}}\int_0^\infty e^{-\rho dt}\left[u\left(\iint c_t(a,z;W_{k,t})g_t(a,z)\,da\,dz\right) - v\left(\int_0^1 \ell_{k,t}\,dk\right)\right]dt,$$

where the union internalizes that its wage policy $W_{k,t}$ affects the household's consumption choice and direct labor supply, subject to the labor packer's demand function. Since union k is small, it takes as given other prices as well as the cross-sectional household distribution.

The first-order condition for $W_{k,t}$ is therefore given by

$$0 = u'(C_t)\int\frac{\partial c_t(a,z;W_{k,t})}{\partial W_{k,t}}g_t(a,z)\,da\,dz + \epsilon^w v'(\ell_t)\frac{\ell_t}{W_t}.$$

By the envelope theorem, we have

$$\frac{\partial c_t(a,z;W_{k,t})}{\partial W_{k,t}} = \frac{1}{P_t}(1+\tau^u)(1-\epsilon^w)z\ell_t.$$

Therefore, we arrive at the aggregate labor supply schedule

$$0 = \frac{1}{P_t}(1+\tau^u)(1-\epsilon^w)\ell_t u'(C_t) + \epsilon^w v'(\ell_t)\frac{\ell_t}{W_t},$$

or simply

$$v'(\ell_t) = \frac{\epsilon^w - 1}{\epsilon^w}(1+\tau^u)w_t u'(C_t),$$

where $w_t = \frac{W_t}{P_t}$.

D.5 Firm Problem

The production function of firm k in sector j at Home is given by

$$y_{jH,t}(k) = A_{jH,t}\ell_{jH,t}(k),$$

and its nominal profits are

$$\Pi_{jH,t}(k) = p_{jH,t}(k)y_{jH,t}(k) - (1-\tau^f_{jH,t})W_t\ell_{jH,t}(k),$$

where the firm takes as given the employment subsidies $\tau^f_{jH,t}$ and the aggregate nominal wage W_t. Nominal marginal cost is therefore

$$MC_{jH,t} = \frac{(1-\tau^f_{jH,t})W_t}{A_{jH,t}}$$

so that we can rewrite profits as $\Pi_{jH,t}(k) = (p_{jH,t} - MC_{jH,t})y_{jH,t}$. Profits and marginal costs in Foreign are defined symmetrically except in terms of the Foreign-currency prices $p^*_{jF,t}(k)$.

Dynamic pricing problem. We assume that firms face a quadratic Rotemberg adjustment cost when changing prices in units of Home currency. Define $\pi_{jH,t}(k) = \dot{p}_{jH,t}(k)/p_{jH,t}(k)$ as the instantaneous rate of price inflation. Then we define the firm's dynamic pricing problem as

$$\max_{\pi_{jH,t}(k)} \int_0^\infty e^{-\rho t}\frac{1}{p_{jH,t}}\left[(1-mc_{jH,t}(k))p_{jH,t}(k)y_{jH,t}(k) - \Lambda(\pi_{jH,t}(k))\right]dt,$$

where the cost of adjusting prices at rate $\pi_{jH,t}(k)$ is given by

$$\Lambda(\pi_{jH,t}(k)) = \frac{\chi_j}{2}\pi_{jH,t}(k)^2 p_{jH,t} y_{jH,t}.$$

The problem for Foreign firms is symmetric.

Lemma 4. *The New Keynesian Phillips curves in Home production sector j characterizes the evolution of sectoral inflation according to*

$$\dot{\pi}_{jH,t} = \rho\pi_{jH,t} - \frac{\epsilon_j}{\chi_j}\left(mc_{jH,t} - \frac{\epsilon_j - 1}{\epsilon_j}\right).$$

The Phillips curve for Foreign-price inflation in Foreign production sector j is given by

$$\dot{\pi}^*_{jF,t} = \rho\pi^*_{jF,t} - \frac{\epsilon^*_j}{\chi^*_j}\left(mc^*_{jF,t} - \frac{\epsilon^*_j - 1}{\epsilon^*_j}\right).$$

Here Foreign inflation is defined as

$$\pi^*_{jF,t} = \frac{\dot{p}^*_{jF,t}}{p^*_{jF,t}} = \frac{\dot{p}_{jF,t}}{p_{jF,t}} - \frac{\dot{\mathcal{E}}_t}{\mathcal{E}_t} = \pi_{jF,t} - \pi^e_t.$$

Proof. Since in equilibrium, all firms are symmetric, we will drop the j indexation for simplicity. Denote $p = P_t(j)$, $P = P_t$, $Y = Y_t$, $\pi = \pi_t$, $\chi = \chi$, $W = W_t$, taking P as given, the firm's problem in recursive form is

$$\rho J(p,t) = \max_{\pi}\left\{\left(\frac{p}{P} - mc\right)\left(\frac{p}{P}\right)^{-\epsilon} Y - \frac{\chi_s}{2}\pi^2 Y + J_p(p,t)p\pi + J_t(p,t)\right\}$$

where J is the corresponding value function of the maximization problem. The first order conditions of the recursive form are given by

$$J_p(p,t)p = \chi\pi Y$$

$$(\rho - \pi)\, J_p(p,t) = -\left(\frac{p}{P} - mc\right)\epsilon\left(\frac{p}{P}\right)^{-\epsilon-1}\frac{Y}{P} + \left(\frac{p}{P}\right)^{-\epsilon}\frac{Y}{P} + J_{pp}(p,t)p\pi + J_{tp}(p,t)$$

In a symmetric equilibrium we will have $p = P$, and hence

$$J_p(p,t) = \frac{\chi\pi Y}{P}$$

$$(\rho - \pi)\, J_p(p,t) = -(1 - mc)\epsilon\frac{Y}{P} + \frac{Y}{P} + J_{pp}(p,t)p\pi + J_{tp}(p,t)$$

Differentiating the first equation with respect to time, we get

$$J_{pp}(p,t)\dot{p} + J_{pt}(p,t) = \frac{\chi Y \dot{\pi}}{P} + \frac{\chi \dot{Y} \pi}{P} - \frac{\chi \pi Y}{P}\frac{\dot{P}}{P}$$

and plugging in the second equation, we get

$$(\rho - \pi)\,\frac{\chi \pi Y}{P} = -(1 - mc)\epsilon\frac{Y}{P} + \frac{Y}{P} + \frac{\chi Y \dot{\pi}}{P} + \frac{\chi \dot{Y} \pi}{P} - \frac{\chi \pi Y}{P}\frac{\dot{P}}{P}$$

Putting it together, we have

$$\dot{\pi}_t = \left(\rho - \frac{\dot{Y}_t}{Y_t}\right)\pi - (mc_t - \frac{\epsilon - 1}{\epsilon})\frac{\epsilon}{\chi}$$

■

D.6 Equilibrium and Implementability

Government. Fiscal policy, monetary policy and tariff policy are exactly as described in the main text. The Taylor rule in continuous time becomes simply

$$i_t = r_{ss} + \phi_\pi \pi_t,$$

where we set $\phi_y = 0$ for simplicity. Foreign's Taylor rule is symmetric.

Interest rates and wages. Household budget constraints depend on the real interest rates $(\boldsymbol{r}, \boldsymbol{r}^*)$ and real wages $(\boldsymbol{w}, \boldsymbol{w}^*)$. Real wages are defined in terms of domestic CPI, so $w_t = \frac{W_t}{P_t}$ and $w_t^* = \frac{W_t^*}{P_t^*}$. Real interest rates are defined as

$$r_t = i_t - \pi_t \qquad \text{and} \qquad r_t^* = i_t^* - \pi_t^*.$$

Equilibrium. The definition of competitive equilibrium is stated in the main text for given initial distributions $g_0(a,z)$ and $g_0^*(a,z)$, which we can derive from given asset holding distributions $\tilde{g}(b_H, b_F, z)$ and $\tilde{g}^*(b_H, b_F, z)$ as explained in Appendix D.3.

We now summarize the equilibrium conditions. At the macro level for Home, we have

$$
\begin{aligned}
y_{jH,t} &= A_{jH,t}\ell_{jH,t} \\
v'(\ell_t) &= \frac{\epsilon^w - 1}{\epsilon^w}(1+\tau^u)w_t u'(C_t) \\
\Pi_t &= \sum_j \left(p_{jH,t}y_{jH,t} - (1-\tau^f_{jH,t})W_t\ell_{jH,t} \right) \\
mc_{jH,t} &= \frac{(1-\tau^f_{jH,t})W_t}{A_{jH,t}p_{jH,t}} \\
\dot{\pi}_{jH,t} &= \rho\pi_{jH,t} - \frac{\epsilon_j}{\chi_j}\left(mc_{jH,t} - \frac{\epsilon_j - 1}{\epsilon_j} \right) \\
r_t &= i_t - \pi_t \\
\ell_t &= \sum_j \ell_{jH,t} \\
\sum_j \tau_{jF,t}p_{jF,t}C_{jF,t} &= T_t + \sum_j \tau^f_{jH,t}W_t\ell_{jH,t} + \tau^u W_t\ell_t \\
i_t &= r_{ss} + \phi_\pi \pi_t \\
y_{jH,t} &= C_{jH,t} + C^*_{jH,t}
\end{aligned}
$$

as well as UIP for exchange rates

$$
1 + i_t = (1+i^*_t)\frac{\dot{\mathcal{E}}_t}{\mathcal{E}_t}.
$$

For Foreign, we have

$$
\begin{aligned}
y^*_{jF,t} &= A^*_{jF,t}\ell^*_{jF,t} \\
v'(\ell^*_t) &= \frac{\epsilon^w - 1}{\epsilon^w}(1+\tau^{u,*})w^*_t u'(C^*_t) \\
\Pi^*_t &= \sum_j p^*_{jF,t}(1 - mc^*_{jF,t})y^*_{jF,t} \\
mc^*_{jF,t} &= \frac{(1-\tau^{f,*}_{jF,t})W^*_t}{A^*_{jF,t}p^*_{jF,t}} \\
\dot{\pi}^*_{jF,t} &= \rho\pi^*_{jF,t} - \frac{\epsilon^*_j}{\chi^*_j}\left(mc^*_{jF,t} - \frac{\epsilon^*_j - 1}{\epsilon^*_j}\right) \\
r^*_t &= i^*_t - \pi^*_t \\
\ell^*_t &= \sum_j \ell^*_{jF,t} \\
0 &= T^*_t + \sum_j \tau^{f,*}_{jF,t}W^*_t\ell^*_{jF,t} + \tau^{u,*}W^*_t\ell^*_t \\
i^*_t &= r^*_{ss} + \phi_\pi\pi^*_t \\
y^*_{jF,t} &= C_{jF,t} + C^*_{jF,t}
\end{aligned}
$$

where

$$
\pi^*_{jF,t} = \frac{\dot{p}^*_{jF,t}}{p^*_{jF,t}} = \frac{\dot{p}_{jF,t}}{p_{jF,t}} - \frac{\dot{\mathcal{E}}_t}{\mathcal{E}_t} = \pi_{jF,t} - \pi^e_t.
$$

We have the aggregation equations for consumption

$$
\begin{aligned}
C_t &= \iint c_t(a,z)g_t(a,z)\,da\,dz \\
C_{j,t} &= \alpha_j\frac{P_{j,t}}{P_t}C_t \\
C_{j\omega,t} &= \theta_{j\omega}\left(\frac{(1+\tau_{j\omega,t})p_{j\omega,t}}{P_{j,t}}\right)^{-\eta_j}C_{j,t}
\end{aligned}
$$

and

$$
\begin{aligned}
C_t^* &= \iint c_t^*(a,z) g_t^*(a,z)\, da\, dz \\
C_{j,t}^* &= \alpha_j^* \frac{P_{j,t}^*}{P_t^*} C_t^* \\
C_{j\omega,t}^* &= \theta_{j\omega}^* \left(\frac{p_{j\omega,t}^*}{P_{j,t}^*} \right)^{-\eta_j^*} C_{j,t}^*
\end{aligned}
$$

And the price indices are

$$
\begin{aligned}
P_t &= \sum_j \alpha_j \log \frac{P_{j,t}}{\alpha_j} \\
P_{j,t} &= \left(\sum_\omega \theta_{j\omega} ((1+\tau_{j\omega,t}) p_{j\omega,t})^{1-\eta_j} \right)^{\frac{1}{1-\eta_j}} \\
P_t^* &= \sum_j \alpha_j^* \log \frac{P_{j,t}^*}{\alpha_j^*} \\
P_{j,t}^* &= \left(\sum_\omega \theta_{j\omega}^* (p_{j\omega,t}^*)^{1-\eta_j^*} \right)^{\frac{1}{1-\eta_j^*}}.
\end{aligned}
$$

Sequence-space representation of the household problem. Notice that the Home household's problem can be written in terms of the sequence of real interest rates $\boldsymbol{r}$ and the earnings function $e_t(z)$, which is given by

$$
\begin{aligned}
e_t(z) &= z w_t \ell_t + z T_t + z \Pi_t \\
&= z w_t \ell_t + z T_t + z \sum_j \frac{p_{jH,t}}{P_t} y_{jH,t} - z \sum_j w_t \ell_{jH,t} + z \sum_j \tau_{jH,t}^f w_t \ell_{jH,t} \\
&= z w_t \ell_t + z \left(\sum_j \tau_{jF,t} p_{jF,t} C_{jF,t} - \sum_j \tau_{jH,t}^f W_t \ell_{jH,t} \right) + z \sum_j \frac{p_{jH,t}}{P_t} y_{jH,t} - z w_t \ell_t + z \sum_j \tau_{jH,t}^f w_t \ell_{jH,t} \\
&= z \left(\sum_j \tau_{jF,t} p_{jF,t} C_{jF,t} - \sum_j \tau_{jH,t}^f W_t \ell_{jH,t} \right) + z \sum_j \frac{p_{jH,t}}{P_t} y_{jH,t} + z \sum_j \tau_{jH,t}^f w_t \ell_{jH,t} \\
&= z \sum_j \frac{\tau_{jF,t} p_{jF,t}}{P_t} C_{jF,t} + z \sum_j \frac{p_{jH,t}}{P_t} y_{jH,t} \\
&= z Y_t + z \tau_t M_t,
\end{aligned}
$$

where we define real GDP as

$$Y_t = \sum_j \frac{p_{jH,t}}{P_t} y_{jH,t}$$

and real aggregate imports as

$$M_t = \sum_j \frac{p_{jF,t}}{P_t} C_{jF,t},$$

and where we impose the restriction of uniform tariffs, $\tau_{jF,t} = \tau_t$, which we currently use in our numerical experiments.

In other words, the Home household problem admits a sequence space representation in terms of

$$c_t(a,z) = \mathcal{C}_t(a,z;\boldsymbol{r},\boldsymbol{Z}),$$

where $Z_t = Y_t + \tau_t M_t$ is aggregate income plus tariff revenue. In other words, we can write the household's savings policy function simply as

$$s_t(a,z) = r_t a + z Z_t - c_t(a,z).$$

To arrive at a sequence-space representation of aggregate Home consumption, we have to be careful about the revaluation effects that show up in the wealth distribution at date 0 as a function of $\mathcal{E}_0$ and P_0. Therefore, we have the aggregate consumption function

$$C_t = \mathcal{C}_t(\boldsymbol{r},\boldsymbol{Z},\mathcal{E}_0,P_0).$$

Working towards implementability. We are therefore left with the Home macro conditions

$$v'\left(\sum_j \frac{y_{jH,t}}{A_{jH,t}}\right) = \frac{\epsilon^w - 1}{\epsilon^w}(1+\tau^u) w_t u'(C_t)$$

$$\dot{\pi}_{jH,t} = \rho \pi_{jH,t} - \frac{\epsilon_j}{\chi_j}\left(\frac{(1-\tau^f_{jH,t}) w_t}{A_{jH,t}} \frac{P_t}{p_{jH,t}} - \frac{\epsilon_j - 1}{\epsilon_j}\right)$$

$$r_t = r_{ss} + (\phi_\pi - 1)\pi_t$$

$$y_{jH,t} = C_{jH,t} + C^*_{jH,t}$$

$$i_t = i^*_t + \pi^e_t$$

and similarly for Foreign

$$v'\left(\sum_j \frac{y^*_{jF,t}}{A^*_{jF,t}}\right) = \frac{\epsilon^w - 1}{\epsilon^w}(1+\tau^u) w^*_t u'(C^*_t)$$

$$\dot{\pi}^*_{jF,t} = \rho \pi^*_{jF,t} - \frac{\epsilon_j}{\chi_j}\left(\frac{(1-\tau^{f,*}_{jF,t}) w^*_t}{A^*_{jF,t}} \frac{P^*_t}{p^*_{jF,t}} - \frac{\epsilon_j - 1}{\epsilon_j}\right)$$

$$r^*_t = r^*_{ss} + (\phi_\pi - 1)\pi^*_t$$

$$y^*_{jF,t} = C_{jH,t} + C^*_{jH,t}$$

Next, we simplify the goods market clearing conditions. For Home, we have

$$C_{jH,t} = (1-\theta_j)\left(\frac{p_{jH,t}}{P_t}\right)^{-\eta_j} \alpha_j \frac{P_{j,t}}{P_t} C_t$$

$$C_{jF,t} = \theta_j \left(\frac{(1+\tau_t) p_{jF,t}}{P_t}\right)^{-\eta_j} \alpha_j \frac{P_{j,t}}{P_t} C_t$$

and for Foreign

$$C^*_{jH,t} = \theta^*_j \left(\frac{p^*_{jH,t}}{P^*_t}\right)^{-\eta^*_j} \alpha^*_j \frac{P^*_{j,t}}{P^*_t} C^*_t$$

$$C^*_{jF,t} = (1-\theta^*_j)\left(\frac{p^*_{jF,t}}{P^*_t}\right)^{-\eta^*_j} \alpha^*_j \frac{P^*_{j,t}}{P^*_t} C^*_t$$

And so therefore we get for Home-produced goods

$$y_{jH,t} = (1-\theta_j)\left(\frac{p_{jH,t}}{P_t}\right)^{-\eta_j} \alpha_j \frac{P_{j,t}}{P_t} C_t + \theta^*_j \left(\frac{p^*_{jH,t}}{P^*_t}\right)^{-\eta^*_j} \alpha^*_j \frac{P^*_{j,t}}{P^*_t} C^*_t$$

and for Foreign-produced goods

$$y^*_{jF,t} = \theta_j \left(\frac{(1+\tau_t) p_{jF,t}}{P_t}\right)^{-\eta_j} \alpha_j \frac{P_{j,t}}{P_t} C_t + (1-\theta^*_j)\left(\frac{p^*_{jF,t}}{P^*_t}\right)^{-\eta^*_j} \alpha^*_j \frac{P^*_{j,t}}{P^*_t} C^*_t$$

We therefore arrive at the following characterization of macro-implementability.

Lemma 5 (Macro Implementability). *Taking as given TFP, policy and aggregate consumption $\boldsymbol{C}$ and $\boldsymbol{C}^*$ in both countries, the allocation $\boldsymbol{y}_{j\omega}$ and prices $(\boldsymbol{p}_{j\omega}, \boldsymbol{w}, \boldsymbol{w}^*)$ form part of a world competitive equilibrium if* *and only if the following conditions are satisfied:*

$$v'\left(\sum_j \frac{y_{jH,t}}{A_{jH,t}}\right) = \frac{\epsilon^w - 1}{\epsilon^w}(1+\tau^u) w_t u'(C_t)$$

$$v'\left(\sum_j \frac{y^*_{jF,t}}{A^*_{jF,t}}\right) = \frac{\epsilon^w - 1}{\epsilon^w}(1+\tau^u) w^*_t u'(C^*_t)$$

$$\dot{\pi}_{jH,t} = \rho\pi_{jH,t} - \frac{\epsilon_j}{\chi_j}\left(\frac{(1-\tau^f_{jH,t})w_t}{A_{jH,t}}\frac{P_t}{p_{jH,t}} - \frac{\epsilon_j - 1}{\epsilon_j}\right)$$

$$\dot{\pi}^*_{jF,t} = \rho\pi^*_{jF,t} - \frac{\epsilon_j}{\chi_j}\left(\frac{(1-\tau^{f,*}_{jF,t})w^*_t}{A^*_{jF,t}}\frac{P^*_t}{p^*_{jF,t}} - \frac{\epsilon_j - 1}{\epsilon_j}\right)$$

$$y_{jH,t} = (1-\theta_j)\left(\frac{p_{jH,t}}{P_t}\right)^{-\eta_j}\alpha_j\frac{P_{j,t}}{P_t}C_t + \theta^*_j\left(\frac{p^*_{jH,t}}{P^*_t}\right)^{-\eta^*_j}\alpha^*_j\frac{P^*_{j,t}}{P^*_t}C^*_t$$

$$y^*_{jF,t} = \theta_j\left(\frac{(1+\tau_t)p_{jF,t}}{P_t}\right)^{-\eta_j}\alpha_j\frac{P_{j,t}}{P_t}C_t + (1-\theta^*_j)\left(\frac{p^*_{jF,t}}{P^*_t}\right)^{-\eta^*_j}\alpha^*_j\frac{P^*_{j,t}}{P^*_t}C^*_t$$

as well as

$$r_{ss} + \phi_\pi \pi_t = r^*_{ss} + \phi_\pi \pi^*_t + \pi^e_t$$

$$P_t = \sum_j \alpha_j \log\frac{P_{j,t}}{\alpha_j}$$

$$P_{j,t} = \left(\sum_\omega \theta_{j\omega}((1+\tau_{j\omega,t})p_{j\omega,t})^{1-\eta_j}\right)^{\frac{1}{1-\eta_j}}$$

$$P^*_t = \sum_j \alpha^*_j \log\frac{P^*_{j,t}}{\alpha^*_j}$$

$$P^*_{j,t} = \left(\sum_\omega \theta^*_{j\omega}(p^*_{j\omega,t})^{1-\eta^*_j}\right)^{\frac{1}{1-\eta^*_j}},$$

where $\pi_{jH,t} = \dot{p}_{jH,t}/p_{jH,t}$ *and* $\pi^*_{jF,t} = \dot{p}_{jF,t}/p_{jF,t}$, *where* $p^*_{j\omega,t} = p_{j\omega,t}/\mathcal{E}_t$, *and where* $\pi^e_t = \dot{\mathcal{E}}_t/\mathcal{E}_t$.

Intuitively, the Phillips curves pin down inflation and therefore prices at the goods level, which in turn gives us the aggregate price indices. The union optimality conditions solve for the equilibrium real wage. UIP solves for the exchange rate. And finally the goods market clearing conditions solve for the aggregate demand for each good and therefore its output.

The above implementability Lemma characterizes the world economy at the macro level in terms of aggregate consumption. It therefore only remains to characterize the determination of aggregate consumption, which we do in the next Lemma.

Lemma 6 (Micro Implementability). *Taking as given initial asset-share distributions $\tilde{g}_0(b_H, b_F, z)$ and $\tilde{g}_0^*(b_H, b_F, z)$, paths of real interest rates $(\boldsymbol{r}, \boldsymbol{r}^*)$, paths of aggregate income plus tariff revenue $(\boldsymbol{Z}, \boldsymbol{Z}^*)$, paths of labor supply $(\boldsymbol{\ell}, \boldsymbol{\ell}^*)$, as well as initial CPI and exchange rates $(P_0, P_0^*, \mathcal{E}_0)$, aggregate consumption sequences $\boldsymbol{C}$ and $\boldsymbol{C}^*$ are consistent with household behavior if and only if the following equations are satisfied:*

$$
\begin{aligned}
\rho V_t(a,z) &= u(c_t(a,z)) - v(\ell_t) + s_t(a,z)\partial_a V_t(a,z) + \lambda(V_t(a,z') - V_t(a,z)) + \partial_t V_t(a,z) \\
\rho V_t^*(a,z) &= u(c_t^*(a,z)) - v(\ell_t^*) + s_t^*(a,z)\partial_a V_t^*(a,z) + \lambda(V_t^*(a,z') - V_t^*(a,z)) + \partial_t V_t^*(a,z) \\
u'(c_t(a,z)) &= \partial_a V_t(a,z) \\
u'(c_t^*(a,z)) &= \partial_a V_t^*(a,z) \\
\partial_t g_t(a,z) &= -\partial_a \Big[s_t(a,z) g_t(a,z)\Big] + \lambda \Big[g_t(a,z') - g_t(a,z)\Big] \\
\partial_t g_t^*(a,z) &= -\partial_a \Big[s_t^*(a,z) g_t^*(a,z)\Big] + \lambda \Big[g_t^*(a,z') - g_t^*(a,z)\Big]
\end{aligned}
$$

where $s_t(a,z) = r_t a + z Z_t - c_t(a,z)$ and $s_t^*(a,z) = r_t^* a + z Z_t^* - c_t^*(a,z)$, as well as the initialization conditions

$$
\begin{aligned}
g_0(a,z) &= \iint_{\mathbb{R}^2} \tilde{g}_0(b_H, b_F, z) \cdot \delta\left(a - \frac{b_H + \mathcal{E}_0 b_F}{P_0}\right) db_H\, db_F \\
g_0^*(a,z) &= \iint_{\mathbb{R}^2} \tilde{g}_0^*(b_H, b_F, z) \cdot \delta\left(a - \frac{b_H/\mathcal{E}_0 + b_F}{P_0^*}\right) db_H\, db_F
\end{aligned}
$$

and the aggregation definitions

$$
\begin{aligned}
C_t &= \iint c_t(a,z) g_t(a,z)\, da\, dz \\
C_t^* &= \iint c_t^*(a,z) g_t^*(a,z)\, da\, dz.
\end{aligned}
$$

D.7 Flexible Price Allocation

In this subsection, we characterize the flexible price (or flexprice) allocation. Monetary policy can implement this allocation in our model when supported by a fiscal policy that sets time-varying corporate employment subsidies in order to equalize labor wedges across sectors. Once sectoral labor wedges are equalized, monetary policy can set interest rates so that the aggregate labor wedge in each economy is 0.

In this case, the macro and micro implementability Lemmas simplify as follows.

Lemma 7 (Implementability: Flexible Prices). *Suppose that monetary and fiscal policy implement the flexprice allocation (or consider the limit $\chi_j \to 0$ for all j). Taking as given initial asset-share distributions $\tilde{g}_0(b_H, b_F, z)$ and $\tilde{g}_0^*(b_H, b_F, z)$, Home tariffs, as well as sequences of TFP, an aggregate allocation*

$(\boldsymbol{y}_{j\omega}, \boldsymbol{y}^*_{j\omega})$, prices $(\boldsymbol{w}, \boldsymbol{w}^*, \boldsymbol{r}, \boldsymbol{r}^*)$, value and policy functions $\{V_t(a,z), V^*_t(a,z), c_t(a,z), c^*_t(a,z)\}$ as well as joint densities $\{g_t(a,z), g^*_t(a,z)\}$ form part of a competitive equilibrium if and only if

$$
\begin{aligned}
v'\left(\sum_j \frac{y_{jH,t}}{A_{jH,t}}\right) &= w_t u'(C_t) \\
v'\left(\sum_j \frac{y^*_{jF,t}}{A^*_{jF,t}}\right) &= w^*_t u'(C^*_t) \\
p_{jH,t} &= \frac{W_t}{A_{jH,t}} \\
p^*_{jF,t} &= \frac{W^*_t}{A^*_{jF,t}} \\
y_{jH,t} &= (1-\theta_j)\left(\frac{p_{jH,t}}{P_t}\right)^{-\eta_j} \alpha_j \frac{P_{j,t}}{P_t} C_t + \theta^*_j \left(\frac{p^*_{jH,t}}{P^*_t}\right)^{-\eta^*_j} \alpha^*_j \frac{P^*_{j,t}}{P^*_t} C^*_t \\
y^*_{jF,t} &= \theta_j \left(\frac{(1+\tau_t) p_{jF,t}}{P_t}\right)^{-\eta_j} \alpha_j \frac{P_{j,t}}{P_t} C_t + (1-\theta^*_j)\left(\frac{p^*_{jF,t}}{P^*_t}\right)^{-\eta^*_j} \alpha^*_j \frac{P^*_{j,t}}{P^*_t} C^*_t
\end{aligned}
$$

where the price indices are defined as before, and

$$
\begin{aligned}
\rho V_t(a,z) &= u(c_t(a,z)) - v(\ell_t) + s_t(a,z)\partial_a V_t(a,z) + \lambda(V_t(a,z') - V_t(a,z)) + \partial_t V_t(a,z) \\
\rho V^*_t(a,z) &= u(c^*_t(a,z)) - v(\ell^*_t) + s^*_t(a,z)\partial_a V^*_t(a,z) + \lambda(V^*_t(a,z') - V^*_t(a,z)) + \partial_t V^*_t(a,z) \\
u'(c_t(a,z)) &= \partial_a V_t(a,z) \\
u'(c^*_t(a,z)) &= \partial_a V^*_t(a,z) \\
\partial_t g_t(a,z) &= -\partial_a \Big[s_t(a,z) g_t(a,z)\Big] + \lambda \Big[g_t(a,z') - g_t(a,z)\Big] \\
\partial_t g^*_t(a,z) &= -\partial_a \Big[s^*_t(a,z) g^*_t(a,z)\Big] + \lambda \Big[g^*_t(a,z') - g^*_t(a,z)\Big]
\end{aligned}
$$

where $s_t(a,z) = r_t a + zZ_t - c_t(a,z)$ and $s^*_t(a,z) = r^*_t a + zZ^*_t - c^*_t(a,z)$, as well as the initialization conditions

$$
\begin{aligned}
g_0(a,z) &= \iint_{\mathbb{R}^2} \tilde{g}_0(b_H, b_F, z) \cdot \delta\left(a - \frac{b_H + \mathcal{E}_0 b_F}{P_0}\right) db_H \, db_F \\
g^*_0(a,z) &= \iint_{\mathbb{R}^2} \tilde{g}^*_0(b_H, b_F, z) \cdot \delta\left(a - \frac{b_H / \mathcal{E}_0 + b_F}{P^*_0}\right) db_H \, db_F
\end{aligned}
$$

D.8 The Ramsey Problem for Optimal Tariffs

We compute optimal tariffs under a utilitarian welfare criterion from the perspective of date 0,

$$W_0 = \iint \Big[u(c_t(a,z)) - v(\ell_t)\Big] g_t(a,z)\, da\, dz.$$

The standard Ramsey problem can therefore be associated with the following Lagrangian:

$$\begin{aligned}
\mathcal{L} = \int_0^\infty e^{-\rho t} \Bigg\{ \iint \Bigg\{ & u(c_t(a,z)) g_t(a,z) - v(\ell_t) g_t(a,z) \\
&+ \phi_t(a,z)\Big[-\rho V_t(a,z) + u(c_t(a,z)) - v(\ell_t) + s_t(a,z)\partial_a V_t(a,z) + \lambda(V_t(a,z') - V_t(a,z)) + \partial_t V_t(a,z)\Big] \\
&+ \phi_t^*(a,z)\Big[-\rho V_t^*(a,z) + u(c_t^*(a,z)) - v(\ell_t^*) + s_t^*(a,z)\partial_a V_t^*(a,z) + \lambda(V_t^*(a,z') - V_t^*(a,z)) + \partial_t V_t^*(a,z)\Big] \\
&+ \chi_t(a,z)\Big[u'(c_t(a,z)) - \partial_a V_t(a,z)\Big] \\
&+ \chi_t^*(a,z)\Big[u'(c_t^*(a,z)) - \partial_a V_t^*(a,z)\Big] \\
&+ \lambda_t(a,z)\Big[-\partial_t g_t(a,z) - \partial_a\Big[s_t(a,z) g_t(a,z)\Big] + \lambda\Big[g_t(a,z') - g_t(a,z)\Big]\Big] \\
&+ \lambda_t^*(a,z)\Big[-\partial_t g_t^*(a,z) - \partial_a\Big[s_t^*(a,z) g_t^*(a,z)\Big] + \lambda\Big[g_t^*(a,z') - g_t^*(a,z)\Big]\Big] \\
&+ \xi(a,z)\Big[-g_0(a,z) + \iint_{\mathbb{R}^2} \tilde{g}_0(b_H, b_F, z)\cdot \delta\Big(a - \frac{b_H + \mathcal{E}_0 b_F}{P_0}\Big)\, db_H\, db_F\Big] \\
&+ \xi^*(a,z)\Big[-g_0^*(a,z) + \iint_{\mathbb{R}^2} \tilde{g}_0^*(b_H, b_F, z)\cdot \delta\Big(a - \frac{b_H/\mathcal{E}_0 + b_F}{P_0^*}\Big)\, db_H\, db_F\Big] \Bigg\}\, da\, dz \\
&+ \dots
\end{aligned}$$

$$
\begin{aligned}
\ldots &+ \vartheta_t\Big[-v'\Big(\sum_j \frac{y_{jH,t}}{A_{jH,t}}\Big) + \frac{\epsilon^w - 1}{\epsilon^w}(1+\tau^u)w_t u'(C_t)\Big] \\
&+ \vartheta_t^*\Big[-v'\Big(\sum_j \frac{y^*_{jF,t}}{A^*_{jF,t}}\Big) + \frac{\epsilon^w - 1}{\epsilon^w}(1+\tau^u)w^*_t u'(C^*_t)\Big] \\
&+ \sum_j \theta_{j,t}\Big[-\dot{\pi}_{jH,t} + \rho\pi_{jH,t} - \frac{\epsilon_j}{\chi_j}\Big(\frac{(1-\tau^f_{jH,t})w_t}{A_{jH,t}}\frac{P_t}{p_{jH,t}} - \frac{\epsilon_j - 1}{\epsilon_j}\Big)\Big] \\
&+ \sum_j \theta^*_{j,t}\Big[-\dot{\pi}^*_{jF,t} + \rho\pi^*_{jF,t} - \frac{\epsilon_j}{\chi_j}\Big(\frac{(1-\tau^{f,*}_{jF,t})w^*_t}{A^*_{jF,t}}\frac{P^*_t}{p^*_{jF,t}} - \frac{\epsilon_j - 1}{\epsilon_j}\Big)\Big] \\
&+ \sum_j \mu_{j,t}\Big[-y_{jH,t} + (1-\theta_j)\Big(\frac{p_{jH,t}}{P_{j,t}}\Big)^{-\eta_j}\alpha_j\frac{P_{j,t}}{P_t}C_t + \theta^*_j\Big(\frac{p^*_{jH,t}}{P^*_{j,t}}\Big)^{-\eta^*_j}\alpha^*_j\frac{P^*_{j,t}}{P^*_t}C^*_t\Big] \\
&+ \sum_j \mu^*_{j,t}\Big[-y^*_{jF,t} + \theta_j\Big(\frac{(1+\tau_t)p_{jF,t}}{P_{j,t}}\Big)^{-\eta_j}\alpha_j\frac{P_{j,t}}{P_t}C_t + (1-\theta^*_j)\Big(\frac{p^*_{jF,t}}{P^*_{j,t}}\Big)^{-\eta^*_j}\alpha^*_j\frac{P^*_{j,t}}{P^*_t}C^*_t\Big] \\
&+ \varphi_t\Big[-r_{ss} - \phi_\pi\pi_t + r^*_{ss} + \phi_\pi\pi^*_t + \pi^e_t\Big] \\
&+ \kappa_t\Big[-P_t + \sum_j \alpha_j \log\frac{P_{j,t}}{\alpha_j}\Big] \\
&+ \kappa^*_t\Big[-P^*_t + \sum_j \alpha^*_j \log\frac{P^*_{j,t}}{\alpha^*_j}\Big] \\
&+ \sum_j \kappa_{j,t}\Big[-P_{j,t} + \Big(\sum_\omega \theta_{j\omega}((1+\tau_{j\omega,t})p_{j\omega,t})^{1-\eta_j}\Big)^{\frac{1}{1-\eta_j}}\Big] \\
&+ \sum_j \kappa^*_{j,t}\Big[-P^*_{j,t} + \Big(\sum_\omega \theta^*_{j\omega}(p^*_{j\omega,t})^{1-\eta^*_j}\Big)^{\frac{1}{1-\eta^*_j}}\Big]
\end{aligned}
$$

where it is understood that $\pi_{jH,t} = \dot{p}_{jH,t}/p_{jH,t}$ and $\pi^*_{jF,t} = \dot{p}_{jF,t}/p_{jF,t}$, where $p^*_{j\omega,t} = p_{j\omega,t}/\mathcal{E}_t$, and where $\pi^e_t = \dot{\mathcal{E}}_t/\mathcal{E}_t$, as well as

$$
C_t = \iint c_t(a,z)g_t(a,z)\,da\,dz
$$

$$
C^*_t = \iint c^*_t(a,z)g^*_t(a,z)\,da\,dz.
$$

D.9 Ramsey Optimality Conditions

We now characterize the Ramsey optimality conditions which define the Ramsey plan.

FOC for distribution. The FOC for the Home distribution defines an HJB for the social value of a household in state (a, z) at date t,

$$\begin{aligned}\rho\lambda_t(a,z) &= \partial_t\lambda_t(a,z) + s_t(a,z)\partial_a\lambda_t(a,z) + \lambda(\lambda_t(a,z') - \lambda_t(a,z)) \\ &\quad + u(c_t(a,z)) - v(\ell_t) + \vartheta_t w_t u''(C_t)c_t(a,z) \\ &\quad + \sum_j \alpha_j \left[(1-\theta_j)\mu_t\left(\frac{p_{jH,t}}{P_t}\right)^{-\eta_j} + \theta_j\mu_t^*\left(\frac{(1+\tau_t)p_{jF,t}}{P_t}\right)^{-\eta_j}\right]\frac{P_{j,t}}{P_t}c_t(a,z)\end{aligned}$$

where the first line on the RHS summarizes the continuation value and the second and third lines represent the social flow payoff. The difference between the private HJB and this social HJB consists of two terms, the last two terms of the social flow payoff.

The FOC for the Foreign distribution, on the other hand, is given by

$$\begin{aligned}\rho\lambda_t^*(a,z) &= \partial_t\lambda_t^*(a,z) + s_t^*(a,z)\partial_a\lambda_t^*(a,z) + \lambda(\lambda_t^*(a,z') - \lambda_t^*(a,z)) \\ &\quad + \vartheta_t^* w_t^* u''(C_t^*)c_t^*(a,z) + \sum_j \alpha_j^*\left[\theta_j^*\mu_t\left(\frac{p_{jH,t}^*}{P_t^*}\right)^{-\eta_j^*} + (1-\theta_j^*)\mu_t^*\left(\frac{p_{jF,t}^*}{P_t^*}\right)^{-\eta_j}\right]\frac{P_{j,t}^*}{P_t^*}c_t^*(a,z)\end{aligned}$$

Here, we can see that the Home Ramsey planner does not place direct welfare weight on Foreign households. Instead, the valuation $\lambda_t^*(a,z)$ is driven entirely by the implementability conditions.

FOC for value function. Next, the FOCs for the Home and Foreign private values are given by

$$\partial_t\phi_t(a,z) = -\mathcal{A}_t^T\phi_t(a,z) + \partial_a\chi_t(a,z)$$

and

$$\partial_t\phi_t^*(a,z) = -(\mathcal{A}_t^*)^T\phi_t^*(a,z) + \partial_a\chi_t^*(a,z),$$

where $\mathcal{A}_t^T$ and $(\mathcal{A}_t^*)^T$ denote the adjoints of the Home and Foreign generators. The multipliers $\phi_t(a,z)$ and $\phi_t^*(a,z)$ solve Kolmogorov forward equations, where $\partial_a\chi_t(a,z)$ and $\partial_a\chi_t^*(a,z)$ act as forcing terms.

FOC for consumption. The last set of optimality conditions at the micro level are the FOCs for consumption. For Home, we have

$$\begin{aligned}-\frac{\chi_t(a,z)}{g_t(a,z)}u''(c_t(a,z)) &= u'(c_t(a,z)) - \partial_a\lambda_t(a,z) + \vartheta_t w_t u''(C_t) \\ &\quad + \sum_j \alpha_j\left[(1-\theta_j)\mu_t\left(\frac{p_{jH,t}}{P_t}\right)^{-\eta_j} + \theta_j\mu_t^*\left(\frac{(1+\tau_t)p_{jF,t}}{P_t}\right)^{-\eta_j}\right]\frac{P_{j,t}}{P_t}\end{aligned}$$

where we used the envelope theorem. And for Foreign, we have

$$-\frac{\chi_t^*(a,z)}{g_t^*(a,z)}u''(c_t^*(a,z)) = u'(c_t^*(a,z)) - \partial_a\lambda_t^*(a,z) + \vartheta_t^* w_t^* u''(C_t^*)$$
$$+\sum_j \alpha_j^*\left[\theta_j^*\mu_t\left(\frac{p_{jH,t}^*}{P_t^*}\right)^{-\eta_j^*} + (1-\theta_j^*)\mu_t^*\left(\frac{p_{jF,t}^*}{P_t^*}\right)^{-\eta_j^*}\right]\frac{P_{j,t}^*}{P_t^*}$$

FOC for sectoral production. For output of good j in Home, we have

$$0 = \mu_{j,t} + \vartheta_t v''(\ell_t)\frac{1}{A_{jH,t}}$$

and for good j in Foreign, we have symmetrically

$$0 = \mu_{j,t}^* + \vartheta_t^* v''(\ell_t^*)\frac{1}{A_{jH,t}^*}$$

FOC for tariffs. For Home import tariff τ_t, the FOC is given by

$$0 = \sum_j \kappa_{j,t}\left(\sum_\omega \theta_{j\omega} p_{j\omega,t}^{1-\eta_j}\right)^{\frac{1}{1-\eta_j}} - \sum_j \mu_{j,t}^*\alpha_j\theta_j\eta_j\left(\frac{(1+\tau_t)p_{jF,t}}{P_t}\right)^{-\eta_j-1}\frac{p_{jF,t}}{P_{j,t}}\frac{P_{j,t}}{P_t}C_t$$
$$+\iint M_t\left[\phi_t(a,z)\partial_a V_t(a,z) + g_t(a,z)\partial_a\lambda_t(a,z)\right] da\, dz$$

In our calibration we set the elasticity of substitution $\eta_j = \eta$ to be uniform across sectors. Also recall that aggregate Home imports are defined as $M_t = \sum_j \frac{p_{jF,t}}{P_t}C_{jF,t}$. Therefore, we have

$$0 = \frac{1}{1+\tau_t}\sum_j \kappa_{j,t}P_{j,t} - \eta\frac{1}{1+\tau_t}\sum_j \mu_{j,t}^*\theta_j\left(\frac{(1+\tau_t)p_{jF,t}}{P_t}\right)^{-\eta}C_{j,t}$$
$$+\iint M_t\left[\phi_t(a,z)\partial_a V_t(a,z) + g_t(a,z)\partial_a\lambda_t(a,z)\right] da\, dz$$

which becomes

$$0 = \frac{1}{1+\tau_t}\sum_j \kappa_{j,t}P_{j,t} - \eta\frac{1}{1+\tau_t}\sum_j \mu_{j,t}^* C_{jF,t}$$
$$+\iint M_t\left[\phi_t(a,z)\partial_a V_t(a,z) + g_t(a,z)\partial_a\lambda_t(a,z)\right] da\, dz$$

And this allows us to solve for tariffs as

$$1+\tau_t = \frac{\eta \sum_j \mu^*_{j,t} C_{jF,t} - \sum_j \kappa_{j,t} P_{j,t}}{\iint M_t \left[\phi_t(a,z) \partial_a V_t(a,z) + g_t(a,z) \partial_a \lambda_t(a,z) \right] da\, dz}$$